

Mirković-Vilonen polytopes and Khovanov-Lauda-Rouquier algebras

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Abstract. We describe how Mirković-Vilonen polytopes arise naturally from the categorification of Lie algebras using Khovanov-Lauda-Rouquier algebras. This gives an explicit description of the unique crystal isomorphism between simple representations of the KLR algebra and MV polytopes.

MV polytopes as defined from the geometry of the affine Grassmannian only make sense for finite dimensional semi-simple Lie algebras, but our construction actually gives a map from the infinity crystal to polytopes in all symmetrizable Kac-Moody algebras. However, to make the map injective and have well-defined crystal operators on the image, we must in general decorate our polytopes with some extra information. We suggest that the resulting **KLR polytopes** are the general-type analogues of MV polytopes.

We give a combinatorial description of the resulting decorated polytopes in all affine cases, and show that this recovers the affine MV polytopes recently defined by Kamnitzer and Baumann and the first author in symmetric affine types. We also briefly discuss the situation beyond affine type.

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INTRODUCTION

Let \mathfrak{g} be a complex semi-simple Lie algebra. In recent years, a number of parametrizations of the crystal $B(-\infty)$ for $U_q^+(\mathfrak{g})$ have been studied. In this paper, we consider the relations between two of these. In the first, the indexing set is the set \mathcal{MV} of Mirković-Vilonen polytopes, as introduced by Anderson [And03] and studied by Kamnitzer [Kam10, Kam07], building on Mirković and Vilonen's work [MV07] on the geometry of the affine Grassmannian. In the second, the indexing set is the set \mathcal{KLR} of simple gradable modules of Khovanov-Lauda-Rouquier (KLR) algebras, as developed by Lauda-Vazirani [LV11] and Kleshchev-Ram [KR]. Since both of these sets index $B(-\infty)$, there is a unique crystal isomorphism between them, but this bijection has not previously been described directly.

Here we give a simple description of this bijection: There is a KLR algebra attached to a symmetrizable Cartan datum and a positive sum $\nu = \sum a_i \alpha_i$ of simple roots. For any two such weights ν_1, ν_2 , there is a natural inclusion $R_{\nu_1} \otimes R_{\nu_2} \hookrightarrow R_{\nu_1 + \nu_2}$. Let P_L be the convex hull of the weights ν' such that $\text{Res}_{\nu', \nu - \nu'}^{\nu} L \neq 0$.

Theorem A *Fix a finite dimensional semi-simple Lie algebra \mathfrak{g} . The map which sends a simple module L over R_{ν} to P_L is the unique crystal isomorphism between \mathcal{KLR} and \mathcal{MV} .*

We feel Theorem A is interesting in its own right, but perhaps more important is the fact that \mathcal{KLR} naturally indexes $B(-\infty)$ for *any* symmetrizable Kac-Moody algebra. Thus, one can try to use the map above to *define* Mirković-Vilonen polytopes outside of finite type. However, one can easily find pairs of non-isomorphic simples with the same polytopes (for example, this occurs in $\widehat{\mathfrak{sl}}_2$ in the weight space 4δ), so the polytopes alone are not enough information to parametrize $B(-\infty)$.

As suggested by Dunlap [Dun10] and developed in [BKT], this problem can be overcome by decorating the edges of P_L with extra information. In the current setting, the most natural data to associate to an edge is a **semi-cuspidal** representation of a smaller KLR algebra (see Definition 2.2). In complete generality, there are many different semi-cuspidal representations that can decorate a given edge, and we do not know a fully combinatorial description of the resulting object. However, in all affine types we do obtain a combinatorial description.

The edges of P_L are always parallel to roots. For edges parallel to real roots, there is only one possible semi-cuspidal representation, so it is safe to leave off the decoration. For now we restrict to the case when \mathfrak{g} is affine of rank $r + 1$, where there is only one minimal imaginary root δ , and this has multiplicity r . In this case, the semi-cuspidal representations that can be associated to a given edge of P_L parallel to δ are naturally indexed by an r -tuple of partitions (Lemma 3.27).

In fact, we can reduce the amount of information even further: As in [BKT], the (possibly degenerate) r -faces of P_L parallel to δ are naturally indexed by the chamber coweights γ of an underlying finite type root system. Denote the face of P_L corresponding to γ by P_L^γ . We in fact decorate P_L with just the data of a partition π^γ for each chamber coweight γ (see Definition 3.28) in such a way that, for any edge E parallel to δ ,

$$(1) \quad E \text{ is a translate of } \left(\sum_{E \subset P_L^\gamma} |\pi^\gamma| \right) \delta.$$

The representation attached to such an edge E is determined in a natural way by $\{\pi^\gamma : E \subset P_L^\gamma\}$.

Define a **decorated affine pseudo-Weyl polytope** to be a pair consisting of

- a polytope P in the root lattice of \mathfrak{g} with all edges parallel to roots and
- a choice of partition π^γ for each chamber coweight γ of the underlying finite type root system which satisfies condition (1) for each edge parallel to δ .

As in finite type, we seek a combinatorial characterization of which decorated pseudo-Weyl polytopes actually occur as P_L for some L .

Notice that for every 2-face F of a decorated pseudo-Weyl polytope, the set of roots α which are parallel to F forms a rank 2 sub-root system Δ_F , which itself is of either finite or affine type. If Δ_F is of affine type, then F is a pseudo-Weyl polytope with two edges parallel to δ , which are of the form $E_\gamma = F \cap P_\gamma$ and $E_{\gamma'} = F \cap P_{\gamma'}$ for unique chamber coweights γ, γ' . We would like to decorate these imaginary edges with π^γ and $\pi^{\gamma'}$, but this is not “decorated” in the sense defined above, since E_γ and $E_{\gamma'}$ are too long and condition (1) fails. Instead, F is the Minkowski sum of the line segment $(\sum_{\gamma: F \subset P^\gamma} |\pi^\gamma|) \delta$ with a decorated polytope \tilde{F} , obtained by shortening E_γ and $E_{\gamma'}$ and decorating them with π^γ and $\pi^{\gamma'}$.

Theorem B *For \mathfrak{g} an affine Lie algebra, the polytopes P_L are precisely the decorated pseudo-Weyl polytopes where every 2-dimensional face F satisfies*

- *If Δ_F is a finite type root system, then F is an MV polytope for that root system (i.e. it satisfies the tropical Plücker relations from [Kam10]).*
- *If Δ_F is of affine type, then \tilde{F} is an MV polytope for that rank 2 affine algebra (either $\widehat{\mathfrak{sl}}_2$ or $A_2^{(2)}$) as defined in [BDKT].*

The description of rank 2 affine MV polytopes in [BDKT] is combinatorial, so Theorem B gives a combinatorial characterization of KLR polytopes in all affine cases.

In [BKT] analogues of MV polytopes were constructed in all symmetric affine types as decorated Harder-Narasimhan polytopes, and it was shown that these are characterized by their 2-faces. Thus Theorem B also allows us to understand the relationship between our decorated polytopes and those defined in [BKT]:

Theorem C *Assume \mathfrak{g} is of affine type with symmetric Cartan matrix. Fix $b \in B(-\infty)$ and let L be the corresponding element of \mathcal{KLR} . Our polytope P_L and the decorated Harder-Narasimhan polytope HN_b from [BKT] have identical underlying polytopes. Furthermore, for each chamber coweight γ in the underlying finite type root system, the partition λ_γ decorating HN_b as defined in [BKT, Sections 1.5 and 7.6] is the transpose of our π^γ .*

It is natural to ask for an intrinsic characterization of the polytopes P_L in the general Kac-Moody case. We do not even have a conjecture for a true combinatorial characterization, since our polytopes are decorated with various semi-cuspidal representations, which at the moment are not well-understood. Some difficulties that come up outside of affine type are discussed in Section 3.7. However, our construction does still satisfy the most basic properties we would expect, as we now summarize (see Corollaries 3.8 and 3.9 for precise statements and proofs).

Theorem D *For \mathfrak{g} an arbitrary symmetrizable Kac-Moody algebra, the map from \mathcal{KLR} to polytopes with edges labeled by representations is injective. Furthermore, for each convex order on roots, the elements of \mathcal{KLR} are parameterized by the tuples of representations of smaller KLR algebras decorating the edges along a corresponding path through the polytope, generalizing the parameterization of crystals in finite type by Lusztig data.*

As we were completing this paper, some independent work on similar problems appeared: McNamara [McN] proved a version of Theorem D in finite type (amongst other theorems on the structure of these representations) and Kleshchev [Kle] gave a generalization of this to affine type. While there was some overlap with the present paper, these other works were focused on a single convex order, rather than giving a description of how different orders interact as we do in Theorems A, B and C.

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1. BACKGROUND

1.1. Crystals. Fix a symmetrizable Kac-Moody algebra and let $\Gamma = (I, E)$ be its Dynkin diagram. We are interested in the crystal $B(-\infty)$ associated with $U^+(\mathfrak{g})$. This is a combinatorial object arising from the theory of crystal bases for the corresponding quantum group (see e.g. [Kas95]). This section contains a brief explanation of the results we need, roughly following [Kas95] and [HK02], to which we refer the reader for details. We start with a combinatorial notion of crystal that includes many examples which do not arise from representations, but which is easy to characterize combinatorially. We then state a theorem of Kashiwara-Saito which allows one to identify if a candidate combinatorial bicrystal is in fact $B(-\infty)$.

Definition 1.1 (see [Kas95, Section 7.2]) A **combinatorial crystal** is a set B along with functions $\text{wt}: B \rightarrow P$ (where P is the weight lattice), and, for each $i \in I$, $\varepsilon_i, \varphi_i: B \rightarrow \mathbb{Z} \cup \{-\infty\}$ and $e_i, f_i: B \rightarrow B \sqcup \{\emptyset\}$, such that

- (i) $\varphi_i(b) = \varepsilon_i(b) + \langle \text{wt}(b), \alpha_i^\vee \rangle$.
- (ii) e_i increases φ_i by 1, decreases ε_i by 1 and increases wt by α_i .
- (iii) $f_i b = b'$ if and only if $e_i b' = b$.
- (iv) If $\varphi_i(b) = -\infty$, then $e_i b = f_i b = \emptyset$.

We often denote a combinatorial crystal simply by B , suppressing the other data.

Definition 1.2 A **lowest weight combinatorial crystal** is a combinatorial crystal which has a distinguished element b_- (the lowest weight element) such that

- (i) The lowest weight element b_- can be reached from any $b \in B$ by applying a sequence of f_i for various $i \in I$.
- (ii) For all $b \in B$ and all $i \in I$, $\varphi_i(b) = \max\{n : f_i^n(b) \neq \emptyset\}$.

Notice that, for a lowest weight combinatorial crystal, all the functions $\varphi_i, \varepsilon_i, \text{wt}$ are determined once the operators f_i and the weight $\text{wt}(b_-)$ of just the lowest weight element are known.

It will be convenient for us to consider a slightly stronger notion, which is less common in the literature:

Definition 1.3 A **bicrystal** is a set B with 2 different crystal structures whose weight functions agree. We will always use the convention of placing a star superscript on all data for the second crystal structure, for example, $e_i^*, f_i^*, \varphi_i^*$, etc.

There is one very important example of a bicrystal for our purposes—the universal lowest weight crystal $B(-\infty)$ along with the usual crystal operators and Kashiwara's

$*$ -crystal operators, which are defined as the conjugates $e_i^* = *e_i*$, $f_i^* = *f_i*$ of the unstarred by an involution $*$: $B(-\infty) \rightarrow B(-\infty)$ (see [Kas93, 2.1.1]). The involution $*$, called the **Kashiwara involution**, is a crystal limit of the bar involution of $U_q^+(\mathfrak{g})$, but it also has a simple combinatorial definition in each of the models of interest to us.

The following is a rewording of [KS97, Proposition 3.2.3] designed to make the roles of the usual crystal operators and the $*$ -crystal operators more symmetric:

Proposition 1.4 *Fix a bicrystal B . Assume (B, e_i, f_i) and (B, e_i^*, f_i^*) are both lowest weight combinatorial crystals with same the same lowest weight element b_- , where the other data is determined by setting $\text{wt}(b_-) = 0$. Assume further that, for all $i \neq j \in I$ and all $b \in B$,*

- (i) $e_i^* e_j(b) = e_j e_i^*(b)$,
- (ii) For all $b \in B$, $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i^\vee \rangle \geq 0$
- (iii) If $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i^\vee \rangle = 0$ then $e_i(b) = e_i^*(b)$,
- (iv) If $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i^\vee \rangle \geq 1$ then $\varphi_i^*(e_i(b)) = \varphi_i^*(b)$ and $\varphi_i(e_i^*(b)) = \varphi_i(b)$.
- (v) If $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i^\vee \rangle \geq 2$ then $e_i e_i^*(b) = e_i^* e_i(b)$.

then $(B, e_i, f_i) \simeq (B, e_i^*, f_i^*) \simeq B(-\infty)$, and $e_i^* = *e_i*$, $f_i^* = *f_i*$, where $*$ is Kashiwara's involution. Furthermore, these conditions are always satisfied by $B(-\infty)$ along with its operators e_i, f_i, e_i^*, f_i^* .

Proof. We simply explain how the statement in [KS97] implies our statement, referring the reader there for specialized notation. Define the map

$$B(-\infty) \rightarrow B(-\infty) \otimes B_i$$

$$b \mapsto (f_i^*)^{\varphi_i^*(b)}(b) \otimes e_i^{\varphi_i^*(b)} b_i.$$

Then one can check that our conditions imply all the conditions from [KS97, Proposition 3.2.3], so that result implies the crystal structure on B defined by e_i, f_i is isomorphic to $B(-\infty)$. The remaining statements then follow from [KS97, Theorem 3.2.2]. \square

We will also make use of Saito's crystal reflections from [Sai94].

Definition 1.5 *Fix $b \in B(-\infty)$ with $\varphi_i^*(b) = 0$. The **Saito reflection** of b is $\sigma_i b = (\tilde{e}_i^*)^{\varepsilon_i(b)} \tilde{f}_i^{\varphi_i(b)} b$. There is also a dual notion of Saito reflection defined by $\sigma_i^*(b) := *(\sigma_i(*b))$ which is defined for those b such that $\varphi_i(b) = 0$*

The operation σ_i does in fact reflect the weight of b by s_i , as the name suggests.

1.2. Convex orders and charges. Fix a symmetrizable Kac-Moody algebra \mathfrak{g} with root system Δ and Cartan subalgebra \mathfrak{h} . Let Δ_+^{\min} be the set of positive roots α such that $x\alpha$ is not a root for any $0 < x < 1$ (this is all positive roots in finite type).

Definition 1.6 A **convex order** is a total order $<$ on Δ_+^{min} such that, given $S, S' \subset \Delta_+^{min}$ with $S \cup S' = \Delta_+^{irr}$ and $\alpha < \alpha'$ for all $\alpha \in S, \alpha' \in S'$, the convex cones $\text{span}_{\mathbb{R}_{\geq 0}} S$ and $\text{span}_{\mathbb{R}_{\geq 0}} S'$ intersect only at the origin.

Notice that any convex order on Δ_+^{min} extends to a preorder on all positive roots, where proportional roots are equivalent.

Definition 1.7 A **charge** is a linear function $c: \mathfrak{h}^* \rightarrow \mathbb{C}$ such that $c(\alpha_i) \neq 0$ for each simple root α_i and such that all $c(\alpha_i)$'s (and thus the images of all positive roots) lie in the upper half-plane.

Every charge defines a preorder $<_c$ on Δ_+^{min} where we define $\alpha <_c \beta$ if and only if $\arg(c(\alpha)) \leq \arg(c(\beta))$, where \arg is the arguments of the complex number (taking a branch cut of \log which does not intersect the upper half plane). If c is generic, this is a total order. Furthermore one can easily verify that, when it is a total order, $<_c$ is always convex.

In general, not all convex orders arise as $<_c$ for a charge c . However, the following will allow us to restrict to considering charges instead of general convex orders in many instances.

Definition 1.8 Fix a pair $(\gamma, <)$, where γ is a positive root, and $<$ is a convex order on Δ_+^{min} . A charge c is said to be $(\alpha, <)$ compatible if, for all $\beta \in \Delta_+$ such that $\gamma - \beta \in \text{span}_{\mathbb{Z}_{\geq 0}} \{\alpha_i\}$, we have $\alpha < \beta$ if and only if $\alpha <_c \beta$ and $\alpha > \beta$ if and only if $\alpha >_c \beta$.

For any fixed α there are only finitely many positive roots β with $\alpha - \beta \in \text{span}_{\mathbb{Z}_{\geq 0}} \{\alpha_i\}$, so it follows easily from the definition of convex order that, for any pair $(\alpha, <)$, there exists a $(\alpha, <)$ -compatible charge.

The following is well known, although we have used a slightly unusual definition of convex order so we include a proof for completeness.

Proposition 1.9 Assume \mathfrak{g} is of finite type. There is a bijection between convex orders on Δ_+ and expressions $\mathbf{i} = i_1 \cdots i_N$ for the longest word w_0 , which is given by sending \mathbf{i} to the order $\alpha_{i_1} < s_{i_1} \alpha_{i_2} < s_{i_1} s_{i_2} \alpha_{i_3} < \cdots < s_{i_1} \cdots s_{i_{N-1}} \alpha_{i_N}$.

Proof. Obviously the order attached to a reduced expression is unique, so we need only show that every convex order is of this form. Given a convex order, there is a unique minimal root and this root is clearly simple, since otherwise it would lie in the span of the roots larger than it. There is a new convex order where $\alpha <' \beta$ if $s_i \alpha < s_i \beta$ or $\beta = \alpha_i$ and $\alpha \neq \alpha_i$. Let $i_1 = i$. Now we define i_2 in the same way from $<'$ and i_3, \dots by iterating this process, until we have done it as many times as there are positive roots. Thus, the list $\alpha_{i_1}, s_{i_1} \alpha_{i_2}, s_{i_1} s_{i_2} \alpha_{i_3}, \dots, s_{i_1} \cdots s_{i_{N-1}} \alpha_{i_N}$ is a complete, irredundant list

of positive roots. This implies that \mathbf{i} is a reduced expression, and thus this order is of the desired form. \square

1.3. Pseudo-Weyl polytopes. We use notation as in the previous subsection.

Definition 1.10 A pseudo-Weyl polytope is a convex polytope P in \mathfrak{h}^* all of whose edges are parallel to roots.

Definition 1.11 For a pseudo-Weyl polytope P , let $\mu_0(P)$ be the vertex of P such that $\langle \mu_0(P), \rho^\vee \rangle$ is minimal, and $\mu^0(P)$ the vertex where this is maximal (these are vertices as for all roots $\langle \alpha, \rho^\vee \rangle \neq 0$).

Lemma 1.12 Fix a pseudo-Weyl polytope P and a convex total order $<$ on Δ_+^{\min} . There is a unique path through the 1-skeleton of P from $\mu_0(P)$ to $\mu^0(P)$ which passes through at most one edge parallel to each root, and these appear in order according to $<$.

Proof. Fix a pseudo-Weyl polytope P , and let $\{\beta_1, \beta_2, \dots, \beta_r\} \in \Delta_+^{\min}$ be the minimal roots that are parallel to edges in P , and order these according to $<$ so that $\beta_1 < \beta_2 < \dots < \beta_r$. Since $<$ is convex, for each $1 \leq k \leq r-1$, one can find $\phi_k \in \mathfrak{h}$ such that $\langle \beta_r, \phi_k \rangle < 0$ for $r \leq k$, and $\langle \beta_r, \phi_k \rangle > 0$ for $r > k$. Let $\phi_0 = \rho^\vee$ and $\phi_r = -\rho^\vee$. Construct a path ϕ_t in coweight space for t ranging from 0 to r by, for $t = k + q$ for $0 \leq q < 1$ letting $\phi_t = (1 - q)\phi_k + q\phi_{k+1}$. As t varies from 0 to r , ϕ_t generically defines a vertex of P , but occasionally defines an edge. The set of edges that come up is the required path. \square

Definition 1.13 Fix a pseudo-Weyl polytope P and a convex order $<$ for each $\alpha \in \Delta_+^{\min}$, define $a_\alpha^<(P)$ to be the unique non-negative integer such that the edge in $P^<$ parallel to α is a translate of $a_\alpha^<(P)\alpha$. We call the collection $\{a_\alpha^<(P)\}$ the **Lusztig data** of P with respect to $<$.

Lemma 1.14 Let P be a pseudo-Weyl polytope and e an edge of P . Then there exists a charge c such that $e \subset P^{<_c}$. In particular, a pseudo-Weyl polytope P is uniquely determined by its Lusztig data with respect to all convex orders $<_c$ coming from charges.

Proof. Since e is an edge of P , there is some functional $\phi \in \mathfrak{h}$ such that the edge e is the locus $e = \{p \in P : \langle p, \phi \rangle \text{ is maximal}\}$. If e is parallel to the root β , this means $\langle \beta, \phi \rangle = 0$, and ϕ may be chosen so that $\langle \beta', \phi \rangle \neq 0$ for all other β' which are parallel to edges of P . For any linear function $f : \mathfrak{h} \rightarrow \mathbb{R}$ such that $f(\Delta_+) \subset \mathbb{R}_+$, define a charge c_f by

$$c_f(p) = \phi(p) + f(p)i.$$

For generic f , c_f satisfies the required conditions. \square

The following should be thought of as a general-type analogue of the fact that, in finite type, each reduced expression for w_0 can be obtained from any other reduced

expression by a finite number of braid moves. In fact, this statement can be generalized to include all convex orders, not just those coming from charges, but we only need the simpler version.

Lemma 1.15 *Let P be a Pseudo-Weyl polytope and c, c' two generic charges. Then there is a sequence of generic charges $c = c_0, c_1, \dots, c_k = c'$ such that $P^{c_0} = P^c$, $P^{c_k} = P^{c'}$, and, for all $k \leq j < k$, P^{c_j} and $P^{c_{j+1}}$ differ by moving around a single 2-face of P in the two possible directions.*

Proof. Let Δ^{res} be the set of root directions that appear as edges in P . For $0 \leq t \leq 1$, let $c_t = (1 - t)c + tc'$. Clearly this is a charge. We can deform c, c' slightly, without changing the order of any of the roots in Δ^{res} , such that

- For all but finitely many t , c_t induces a total order on Δ^{res} .
- For those t where c_t does not induce a total order, there is exactly one argument $0 < a_t < \pi$ such that there are more than one root in Δ^{res} with argument a_t . Furthermore, the span of the roots with argument a_t is 2 dimensional.

Denote the values of t where c_t does not induce a total order by s_1, \dots, s_{k-1} . Fix t_1, \dots, t_k with

$$0 = t_0 < s_1 < t_1 < s_2 < \dots < t_{k-1} < s_{k-1} < t_k = 1.$$

one can check that $c_j = c_{t_j}$ is the required sequence. \square

1.4. Finite type MV polytopes. In finite type, Anderson [And03] and Kamnitzer [Kam10, Kam07] developed a realization of $B(-\infty)$ where the underlying set consists of Mirković-Vilonen (MV) polytopes. These are certain polytopes in weight space; the only facts about these we will need are certain characterization theorems.

First we consider rank 2. In this case there are exactly two convex orders, determined uniquely by the choice of the order on the simple roots α_1 and α_2 . We fix the order of the form

$$\alpha_1 = \beta_1 < \beta_2 < \dots < \beta_{N-1} < \beta_N = \alpha_2$$

A pseudo-Weyl polytope is equivalent to the data of two Lusztig data (a^1, \dots, a^N) and $(\bar{a}^1, \dots, \bar{a}^N)$ describing the two sides of the polytope, where (a^1, \dots, a^N) is the Lusztig data for the chosen order.

The following is shown in [MT, Theorem 3.8 and Remark 3.10]. It can also be proven directly by a straightforward inductive argument.

Proposition 1.16 *Assume \mathfrak{g} finite type and rank 2. There is a unique map from $B(-\infty)$ to pseudo-Weyl polytopes such that*

- (i) $\text{wt}(b) = \mu^0(P_b) - \mu_0(P_b)$.
- (ii) $a^1(P_{e_1b}) = a^1(P_b) + 1$, and, for all other k , $a^k(P_{e_1b}) = a^k(P_b)$. Similarly $\bar{a}^1(P_{e_0b}) = \bar{a}^1(P_b) + 1$, and, for all other k , $\bar{a}^k(P_{e_0b}) = \bar{a}^k(P_b)$.

- (iii) If $\varphi_1(P_b) = 0$, then for all $k > 1$, $a^k(P_b) = \bar{a}^{k-1}(P_{\sigma_1(b)})$ and $\bar{a}^k(P_{\sigma_1(b)}) = 0$. Similarly if $\varphi_2(P_b) = 0$, then for all $k > 1$, $\bar{a}^k(P_b) = a^{k-1}(P_{\sigma_2(b)})$ and $a^k(P_{\sigma_2(b)}) = 0$.

Here σ_1, σ_2 are Saito reflections. This map is the unique bicrystal isomorphism between $B(-\infty)$ and the set of MV polytopes. \square

We also need the following standard facts about MV polytopes:

Theorem 1.17 ([Kam10, Theorem D]) *The MV polytopes are exactly those pseudo-Weyl polytopes such that all 2-faces are MV polytopes for the corresponding rank 2 root system.* \square

Theorem 1.18 ([Kam10, 4.2]) *An MV polytope is uniquely determined by its Lusztig data with respect to any convex order on positive roots.* \square

1.5. Rank 2 affine MV polytopes. We briefly review the MV polytopes associated to the affine root systems $\widehat{\mathfrak{sl}}_2$ and $A_2^{(2)}$ in [BDKT], and recall a characterization of the resulting polytopes developed in [MT]. These constructions are discussed in more detail in the references mentioned, and we refer the reader there for more details.

The $\widehat{\mathfrak{sl}}_2$ and $A_2^{(2)}$ root systems correspond to the affine Dynkin diagrams

$$\widehat{\mathfrak{sl}}_2 : \begin{array}{ccc} \bullet & \longleftrightarrow & \bullet \\ 0 & & 1 \end{array}, \quad A_2^{(2)} : \begin{array}{ccc} \bullet & \Longleftarrow\!\!\!\! \Longleftarrow & \bullet \\ 0 & & 1 \end{array}.$$

The corresponding symmetrized Cartan matrices are

$$\widehat{\mathfrak{sl}}_2 : N = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad A_2^{(2)} : N = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}.$$

Denote the simple roots by α_0, α_1 , where in the case of $A_2^{(2)}$ the short root is α_0 . Define $\delta = \alpha_0 + \alpha_1$ for $\widehat{\mathfrak{sl}}_2$ and $\delta = 2\alpha_0 + \alpha_1$ for $A_2^{(2)}$.

The dual Cartan subalgebra \mathfrak{h}^* of \mathfrak{g} is a three dimensional vector space containing α_0, α_1 . This has a standard non-degenerate bilinear form (\cdot, \cdot) such that $(\alpha_i, \alpha_j) = N_{i,j}$. Notice that $(\alpha_0, \delta) = (\alpha_1, \delta) = 0$. Fix fundamental coweights ω_0, ω_1 which satisfy $(\alpha_i, \omega_j) = \delta_{i,j}$, where we are identifying coweight space with weight space using (\cdot, \cdot) .

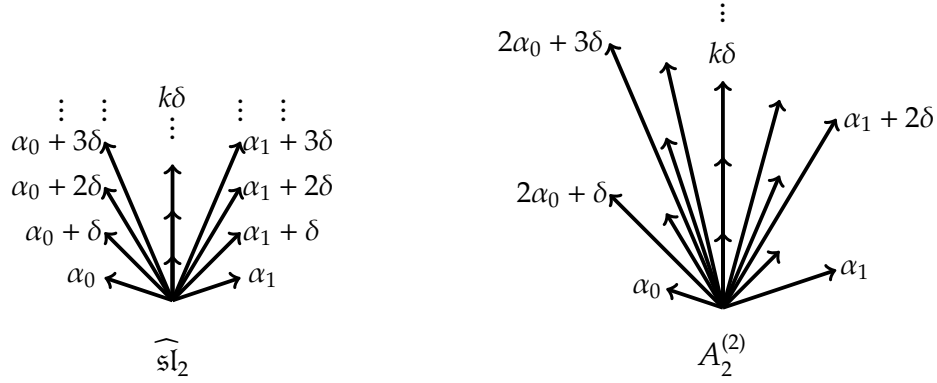
The set of positive roots for $\widehat{\mathfrak{sl}}_2$ is

$$(2) \quad \{\alpha_0, \alpha_0 + \delta, \alpha_0 + 2\delta, \dots\} \sqcup \{\alpha_1, \alpha_1 + \delta, \alpha_1 + 2\delta, \dots\} \sqcup \{\delta, 2\delta, 3\delta, \dots\},$$

where the first two families consist of real roots and the third family of imaginary roots. The set of positive roots for $A_2^{(2)}$ is

$$(3) \quad \Delta_{re}^+ = \{\alpha_0 + k\delta, \alpha_1 + 2k\delta, \alpha_0 + \alpha_1 + k\delta, 2\alpha_0 + (2k+1)\delta \mid k \geq 0\} \quad \text{and} \quad \Delta_{im}^+ = \{k\delta \mid k \geq 1\},$$

where Δ_{re}^+ consists of real roots and Δ_{im}^+ of imaginary roots. We draw these as



Definition 1.19 Label the positive real roots by r_k, r^k for $k \in \mathbb{Z}_{>0}$ by:

- For $\widehat{\mathfrak{sl}}_2$: $r_k = \alpha_1 + (k-1)\delta$ and $r^k = \alpha_0 + (k-1)\delta$.
- For $A_2^{(2)}$:

$$r_k = \begin{cases} \tilde{\alpha}_1 + (k-1)\tilde{\delta} & \text{if } k \text{ is odd,} \\ \tilde{\alpha}_0 + \tilde{\alpha}_1 + \frac{k-2}{2}\tilde{\delta} & \text{if } k \text{ is even,} \end{cases} \quad r^k = \begin{cases} \tilde{\alpha}_0 + \frac{k-1}{2}\tilde{\delta} & \text{if } k \text{ is odd,} \\ 2\tilde{\alpha}_0 + (k-1)\tilde{\delta} & \text{if } k \text{ is even.} \end{cases}$$

There are exactly two convex orders on Δ_+^{min} : The order $<_+$

$$r_1 <_+ r_2 <_+ \dots <_+ \delta <_+ \dots <_+ r^2 <_+ r^1,$$

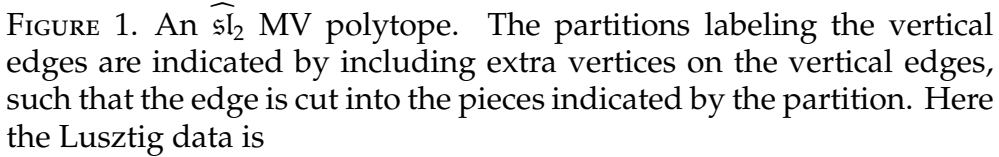
and the reverse of this order, which we denote by $<_-$.

Definition 1.20 A rank 2 affine decorated pseudo-Weyl polytope is a pseudo-Weyl polytope along with a choice of two partitions $a_\delta = (\lambda_1 \geq \lambda_2 \geq \dots)$ and $\bar{a}_\delta = (\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \dots)$ such that $\mu^\infty - \mu_\infty = |a_\delta|\delta$, $\bar{\mu}^\infty - \bar{\mu}_\infty = |\bar{a}_\delta|\delta$. Here $|a_\delta| = \lambda_1 + \lambda_2 + \dots$ and $|\bar{a}_\delta| = \bar{\lambda}_1 + \bar{\lambda}_2 + \dots$.

Definition 1.21 The right Lusztig data of a decorated pseudo-Weyl polytope P is the refinement $\mathbf{a} = (a_\alpha)_{\alpha \in \tilde{\Delta}_+}$ of the Lusztig data from Section 1.3 with respect to $<_+$ (which records the lengths of the edges parallel to each root up one side of P), where, for $\alpha \neq \delta$, $a_\alpha = a_\alpha^{<+}(P)$, and a_δ is as in Definition 1.20. Similarly the left Lusztig data is $\bar{\mathbf{a}} = (\bar{a}_\alpha)_{\alpha \in \tilde{\Delta}_+}$ where, for $\alpha \neq \delta$, $\bar{a}_\alpha = \bar{a}_\alpha^{<-}(P)$, and \bar{a}_δ is as in Definition 1.20.

In [BDKT], the first author and collaborators combinatorially define a set \mathcal{MV} of decorated pseudo-Weyl polytopes, which they call **rank 2 affine MV polytopes**. We will not need the details of this construction, but will instead use the following result from [MT]. Assume \mathfrak{g} is of type $\widehat{\mathfrak{sl}}_2$ or $A_2^{(2)}$. Let

$$\ell_0 = \begin{cases} 2 & |\alpha_0| < |\alpha_1| \\ 1 & |\alpha_0| \geq |\alpha_1| \end{cases} \quad \ell_1 = \begin{cases} 2 & |\alpha_1| < |\alpha_0| \\ 1 & |\alpha_1| \geq |\alpha_0| \end{cases}$$


$$a_1 = 2, a_2 = 1, a_3 = 1, \lambda = (9, 2, 1, 1), a^3 = 1, a^1 = 1, \\ \bar{a}_1 = 1, \bar{a}_2 = 2, \bar{a}_3 = 1, \bar{a}_4 = 1, \bar{\lambda} = (2, 1, 1), \bar{a}^4 = 1, \bar{a}^2 = 1, \bar{a}^1 = 5, \\ \text{and all other } a_k, a^k, \bar{a}_k, \bar{a}^k \text{ are 0.}$$

Theorem 1.22 [MT, Theorem 3.10] *There is a unique map $b \rightarrow P_b$ from $B(-\infty)$ to type \mathfrak{g} decorated pseudo-Weyl polytopes (considered up to translation) such that, for all $b \in B(-\infty)$, the following hold.*

- (i) $\text{wt}(b) = \mu^0(P_b) - \mu_0(P_b)$.
- (ii.1) $a_{\alpha_0}(P_{e_0b}) = a_{\alpha_0}(P_b) + 1$, and for all other root directions $a_{\alpha}(P_{e_0b}) = a_{\alpha}(P_b)$;
- (ii.2) $\bar{a}_{\alpha_1}(P_{e_1b}) = \bar{a}_{\alpha_1}(P_b) + 1$, and for all other root directions $\bar{a}_{\alpha}(P_{e_1b}) = \bar{a}_{\alpha}(P_b)$;
- (ii.3) $a_{\alpha_1}(P_{e_1^*b}) = a_{\alpha_1}(P_b) + 1$, and for all other root directions $a_{\alpha}(P_{e_1^*b}) = a_{\alpha}(P_b)$;
- (ii.4) $\bar{a}_{\alpha_0}(P_{e_0^*b}) = \bar{a}_{\alpha_0}(P_b) + 1$, and for all other root directions $\bar{a}_{\alpha}(P_{e_0^*b}) = \bar{a}_{\alpha}(P_b)$.
- (iii.1) Let σ_0, σ_1 denote Saito's reflections. If $a_{\alpha_0}(P_b) = 0$, then for all $\alpha \neq \alpha_0$, $a_{\alpha}(P_b) = \bar{a}_{s_0(\alpha)}(P_{\sigma_0(b)})$ and $\bar{a}_{\alpha_0}(P_{\sigma_0(b)}) = 0$;
- (iii.2) if $\bar{a}_{\alpha_1}(P_b) = 0$, then for all $\alpha \neq \alpha_1$, $\bar{a}_{\alpha}(P_b) = a_{s_1(\alpha)}(P_{\sigma_1(b)})$ and $a_{\alpha_1}(P_{\sigma_1(b)}) = 0$;
- (iii.3) if $\bar{a}_{\alpha_0}(P_b) = 0$, then for all $\alpha \neq \alpha_0$, $\bar{a}_{\alpha}(P_b) = a_{s_0(\alpha)}(P_{\sigma_0^*(b)})$ and $a_{\alpha_0}(P_{\sigma_0^*(b)}) = 0$;
- (iii.4) if $a_{\alpha_1}(P_b) = 0$, then for all $\alpha \neq \alpha_1$, $a_{\alpha}(P_b) = \bar{a}_{s_1(\alpha)}(P_{\sigma_1^*(b)})$ and $\bar{a}_{\alpha_1}(P_{\sigma_1^*(b)}) = 0$.

(iv) If $a_\beta(P_b) = 0$ for all real roots β and $a_\delta(P_b) = \lambda \neq 0$ then:

$$\bar{a}_{\alpha_1}(P_b) = \ell_1 \lambda_1; \quad \bar{a}_\delta(P_b) = \lambda \setminus \lambda_1; \quad \bar{a}_{\alpha_0}(P_b) = \ell_0 \lambda_1;$$

$$\bar{a}_\beta(P_b) = 0 \text{ for all other } \beta \in \tilde{\Delta}_+.$$

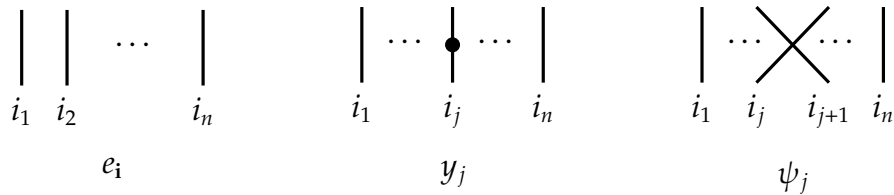
The image of this map is exactly MV as defined in [BDKT]. \square

1.6. Khovanov-Lauda-Rouquier algebras. In this section, we recall the basic facts about the **Khovanov-Lauda-Rouquier algebras** (sometimes called quiver Hecke algebras) attached to the Lie algebra \mathfrak{g} as defined for Kac-Moody algebras in [KL09, Rou], and extended to the case of Borcherds algebras in [KOP].

This is an algebra R (we will omit the Lie algebra \mathfrak{g}) built out of generic string diagrams, i.e. immersed 1-dimensional submanifolds of \mathbb{R}^2 whose boundary lies on the lines $y = 0$ and $y = 1$, where the projection of each component (not a connected component, but an immersed copy of the interval) projects isomorphically to $[0, 1]$ under the projection to the y -axis. These are assumed to have

- no self-intersections
- no components where both ends connect to $y = 0$ or both to $y = 1$
- no points with that lie on 3 or more components
- no points where components intersect non-transversely.

Each string is labeled with a simple root of the corresponding Kac-Moody algebra, and each string is allowed to carry dots at any point where it does not intersect another (but with only finitely many dots in each diagram). All of these diagrams are only considered up to isotopy preserving all these conditions. We will use e_i to denote the idempotent element which is straight lines labeled with the sequence $\mathbf{i} = (i_1, \dots, i_n)$, y_i to denote the diagram which is just straight lines with a dot on the i th strand and ψ_i to denote the crossing of the i and $i + 1$ st strand with no other decorations.



Fix a field \mathbb{k} . The \mathbb{k} -linear combinations of these diagrams form an algebra where ab is formed by stacking the diagram of a on that of b , shrinking vertically by a factor of 2 and smoothing kinks; if the labels of the line $y = 0$ for a and $y = 1$ for b cannot be isotoped to match, the product is 0.

In order to arrive at the algebra R , we must impose relations. All of these relations are local in nature, that is, if we recognize a small piece of a diagram which looks like the LHS of a relation, we can replace it with the RHS, leaving the rest unchanged.

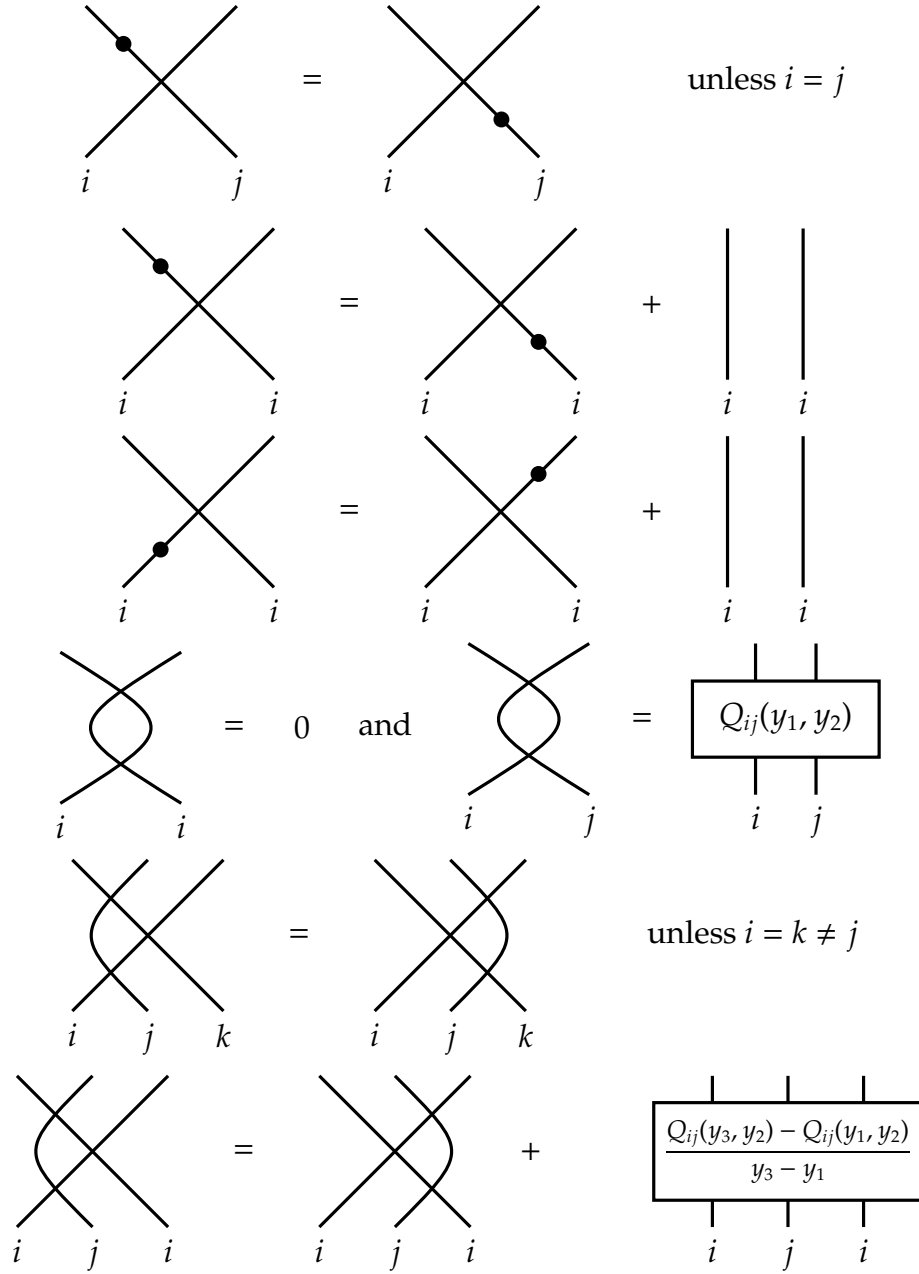


FIGURE 2. The relations of the KLR algebra. These relations are insensitive to labeling of the plane.

These relations depend on a choice of a polynomial $Q_{ij}(u, v) \in \mathbb{k}[u, v]$ for $i \neq j$ indexing simple roots (by convention $Q_{ii} = 0$). Let $C = (c_{ij})$ be the Cartan matrix of \mathfrak{g} and d_i be coprime integers so that $d_j c_{ij} = d_i c_{ji}$. We assume each polynomial is homogeneous of degree $\langle \alpha_i, \alpha_j \rangle = -2d_j c_{ij} = -2d_i c_{ji}$ when u is given degree $2d_i$ and v degree $2d_j$. We will

always assume that the leading order of Q_{ij} in u is $-c_{ji}$, and that $Q_{ij}(u, v) = Q_{ji}(v, u)$. The relations we impose are shown in Figure 2.

In [LV11, 1.1.4], Lauda and Vazirani define an automorphism $\sigma: R \rightarrow R$ which up to sign reflects the diagrams through the vertical axis. We let M^σ denote the twist of an R -module by this automorphism.

While some other aspects of the representation theory of R are quite sensitive to the choice of \mathbb{k} and Q_{ij} (for example, the dimensions of simple modules), none of the theorems we prove will depend on it; the reader is free to imagine that we have chosen their favorite field and worked with it throughout.

Since the diagrams allowed in R never change the sum of the simple roots labeling the strands, it breaks up as a direct sum of algebras $R \cong \bigoplus_\nu R_\nu$, where that sum is fixed to be $\nu \in \mathfrak{h}^*$. In particular, for any simple R -module L , there is a unique ν such that $R_\nu \cdot L = L$. We call this the weight of L . We let \mathcal{L}_i denote the unique 1-dimensional simple of R_{α_i} .

We will only ever consider modules over R_ν which are finite dimensional and on which all the y_i 's act nilpotently; for simple modules, this is equivalent to Lauda and Vazirani's condition that their modules are gradable. If we let $K_0^q(R)$ denote the complexified Grothendieck group of the category of graded finite dimensional modules over R , then we have an isomorphism $K_0^q(R) \cong U_q(\mathfrak{n})$. The simple modules over R give a basis of $U_q(\mathfrak{n})$, which while not necessarily a crystal basis, still carries a crystal structure:

Proposition 1.23 ([LV11, Section 5.1]) *The set \mathcal{KLR} of isomorphism classes of gradable simple modules over R carry a crystal structure with operators defined by*

$$\tilde{e}_i L = \text{cosoc}(L \circ \mathcal{L}_i) \quad \tilde{f}_i L = \text{cosoc}(\text{Hom}_{R_{\nu-\alpha_i} \otimes R_{\alpha_i}}(R_{\nu-\alpha_i} \otimes \mathcal{L}_i, \text{Res}_{\nu-\alpha_i, \alpha_i}^\nu L)),$$

and this crystal is isomorphic to $B(-\infty)$. The map $(-)^{\sigma}: \mathcal{KLR} \rightarrow \mathcal{KLR}$ is intertwined with the Kashiwara involution of $B(-\infty)$.

Remark 1.24 *Our conventions are dual to those of [LV11], since we consider $B(-\infty)$ rather than $B(\infty)$.*

Remark 1.25 *In [LV11], the operator \tilde{f}_i was actually defined as a socle, not a cosocle; however, as noted by Khovanov and Lauda [KL09, §3.2], all simple modules over the KLR algebra are self-dual, and $\text{Hom}_{R_{\nu-\alpha_i} \otimes R_{\alpha_i}}(R_{\nu-\alpha_i} \otimes \mathcal{L}_i, \text{Res}_{\nu-\alpha_i, \alpha_i}^\nu -)$ commutes with duality, so its socle and cosocle when applied to a simple module are isomorphic.*

There are orthogonal idempotents e_i attached to every word in the simple roots of \mathfrak{g} , and we consider the “character”

$$\text{ch}(M) = \sum_{\mathbf{i}} \dim_q(e_{\mathbf{i}} M) \cdot w[\mathbf{i}]$$

as an element of \mathcal{F} , the free $\mathbb{C}[q, q^{-1}]$ -module generated by words in the nodes of the Dynkin diagram. By [KL09, 2.20], we have the fundamental relation

$$\text{ch}(M_1 \circ M_2) = \text{ch}(M_1) \circ \text{ch}(M_2).$$

In fact, we have a commutative diagram

$$\begin{array}{ccc} K_0^q(R) \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{\text{ch}} & \mathcal{F} \\ & \searrow \Phi & \nearrow \\ & U_q(\mathfrak{n}) & \end{array}$$

where \mathcal{F} carries the shuffle product and Φ is the usual shuffle expansion by [KL09, 2.20].

2. KLR ALGEBRAS AND LUSZTIG DATA

Kleshchev and Ram's work [KR] studying simple representations of KLR algebras in terms of Lyndon word combinatorics allows one to construct a Lusztig datum for each KLR module with respect to any convex order which arises from a lexicographic order on Lyndon words. We now extend this to obtain a Lusztig datum for any convex order. In general, we can no longer use the same type of combinatorics on words that they develop, and instead our main tool is the notion of a cuspidal representation with respect to a “charge”.

2.1. Cuspidal representations. Let $\mathbf{i} = i_1 \cdots i_n$ be a word in the nodes of the Dynkin diagram and let $\alpha_{\mathbf{i}} = \sum_{k=1}^n \alpha_{i_k}$. Fix a charge c , and consider the preorder $<$ on positive elements of the root lattice induced by taking arguments with respect to this charge, as in Section 1.2.

Definition 2.1 *The **top** of a word \mathbf{i} is the maximal element which appears as the sum of a proper left prefix of the word; that is*

$$\text{top}(\mathbf{i}) = \max_{1 \leq j < n} \alpha_{i_1 \cdots i_j}.$$

*We call a word in the simple roots **c-cuspidal** if $\text{top}(\mathbf{i}) < \alpha_{\mathbf{i}}$ and **c-semi-cuspidal** if $\text{top}(\mathbf{i}) \leq \alpha_{\mathbf{i}}$*

Geometrically, we can visualize our word as a path in the weight lattice, and then picture its image in the complex plane under c . A word is c -cuspidal if this path stays strictly clockwise of the line from the beginning to the end of the word and c -semi-cuspidal if stays weakly clockwise of this line, as shown in Figure 3.

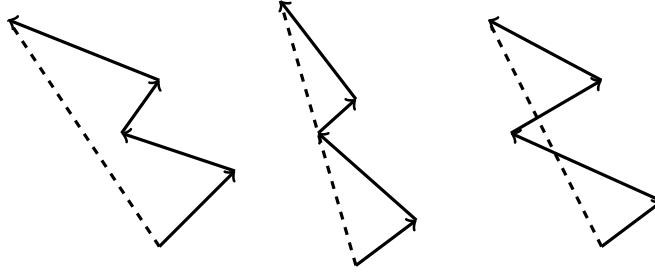


FIGURE 3. Examples of c -cuspidal, c -semi-cuspidal, and non- c -semi-cuspidal paths.

Definition 2.2 The **top** of a module over R is the maximum among the tops of all \mathbf{i} such that $e_{\mathbf{i}}M \neq 0$. We call a simple module over R_v **cuspidal** if $\text{top}(L) < v$, and **semi-cuspidal** if $\text{top}(L) \leq v$.

Obviously, a simple representation is (semi-)cuspidal if and only if all words which appear in its character are (semi-)cuspidal. Even if c is not generic and only induces a pre-order, the following theorem still holds.

Theorem 2.3 Consider L_1, \dots, L_h are semi-cuspidal representations with $v_k = \text{wt}(L_k)$. If $v_1 > \dots > v_h$, then the module $L_1 \circ \dots \circ L_h$ has a unique simple quotient. Furthermore, every simple appears this way for a unique sequence of semi-cuspidal representations.

In order to prove this, we introduce a more general compatibility condition on representations:

Definition 2.4 We call an h -tuple (L_1, \dots, L_h) **unmixing** if

$$\text{Res}_{v_1, \dots, v_h}^{v_1 + \dots + v_h}(L_1 \circ \dots \circ L_h) = L_1 \boxtimes \dots \boxtimes L_h.$$

Lemma 2.5 If (L_1, \dots, L_h) is an unmixing h -tuple, then $L_1 \circ \dots \circ L_h$ has a unique simple quotient.

Proof. Let e denote the idempotent in R_v projecting to $\text{Res}_{v_1, \dots, v_h}^{v_1 + \dots + v_h}(-)$. Then $L_1 \circ \dots \circ L_h$ is generated by any non-zero vector in the image of e ; thus, a submodule $M \subset L_1 \circ \dots \circ L_h$ is proper if and only if it is killed by e . It follows that the sum of any two proper submodules is still killed by e , and thus again proper. There is thus a unique maximal proper submodule of $L_1 \circ \dots \circ L_h$, which is to say this module has a simple cosocle. This establishes the theorem. \square

Proof of Theorem 2.3. We will establish this theorem by showing that the h -tuple (L_1, \dots, L_h) is unmixing.

Consider such an induction of cuspidal representations; assume that \mathbf{i} is a sequence of nodes in the Dynkin diagram such that $e_{\mathbf{i}}(L_1 \circ \cdots \circ L_h) \neq 0$. By [KL09, 2.19], \mathbf{i} must be a shuffle of sequences $\mathbf{j}_1, \dots, \mathbf{j}_h$ such that $e_{\mathbf{j}_k} L_k \neq 0$. The proof of [KL09, 2.18] shows that if the only such shuffles which occur are simply concatenating in order, then $e_{\mathbf{i}}(L_1 \circ \cdots \circ L_h) = e_{\mathbf{j}_1} L_1 \boxtimes \cdots \boxtimes e_{\mathbf{j}_h} L_h$.

Now, assume that the first $\rho^\vee(v_1)$ elements of \mathbf{i} sum to v_1 , the next $\rho^\vee(v_2)$ elements sum to v_2 , etc. Call these groups of elements summing to v_k the **blocks** of \mathbf{i} . Consider a sequence $\mathbf{j}_1, \dots, \mathbf{j}_h$ such that $e_{\mathbf{j}_k} L_k \neq 0$.

For each $1 \leq k, g \leq h$, let v_k^g be the sum over the elements of \mathbf{j}_g which are shuffled into the k th block of \mathbf{i} . For each g , v_1^g is the sum of the roots in a prefix of a word in the character of L_g . Thus by semi-cuspidality, $v_1^1 \leq v_1$ and, for $g > 1$, we have $v_1^g \leq v_k < v_1$.

Clearly $v_1^1 + \cdots + v_1^h = v_1$, so this is only possible if $v_1^1 = v_1$. Applying this argument inductively, we see that $\mathbf{i} = \mathbf{j}_1 \cdots \mathbf{j}_h$, and the only shuffle which can arrive here is the trivial one. It follows that (L_1, \dots, L_h) is unmixing; Lemma 2.5 shows immediately that this induction has simple cosocle.

Now fix a simple L . Consider the convex hull in \mathbb{C} of $\{c(v') \mid \text{Res}_{v', v''}^v L \neq 0\}$. This polytope has two distinguished sides separated by the vertices at 0 and $c(v)$. By the construction of the polytope we can find v_1, \dots, v_h such that the vertices on the left side (i.e. with greater argument than $c(v)$) are exactly of the form $v_1 + \cdots + v_\ell$ for various ℓ , and such that $L|_{R(v_1) \boxtimes \cdots \boxtimes R(v_h)} \neq 0$. Obviously, the restriction of $\text{Res}_{v_1, \dots, v_h}^v L$ only has composition factors which are outer tensor products of semi-cuspidals, or we could find a new vertex in our polytope. Fix any simple submodule $L_1 \boxtimes \cdots \boxtimes L_h$ of $\text{Res}_{v_1, \dots, v_h}^v L$. By Frobenius reciprocity, we have a non-zero map $L_1 \circ \cdots \circ L_h \rightarrow L$, which is surjective by the simplicity of L . As above $L_1 \circ \cdots \circ L_h$ has a unique simple quotient, so this must be L . \square

Definition 2.6 For a fixed charge c and simple L , we call the associated simples (L_1, \dots, L_h) for a fixed charge the **c-semi-cuspidal decomposition** of L .

Corollary 2.7 Fix a charge c . The number of c -semi-cuspidal representations of weight v is the number of distinct ways of writing v as a sum of roots α which all satisfy $\arg c(\alpha) = \arg c(v)$. In imaginary root spaces of multiplicity greater than 1 one chooses a basis for this root space and calls those the roots).

Proof. We proceed by induction on $\rho^\vee(v)$. If v is a simple root, then the statement is obvious, providing the base case.

In general, the dimension of $U(\mathfrak{n})_v$ is the number of ways of writing v as a sum of multiples of positive roots, so this is the number of simple representations of R_v . By the inductive assumption and Theorem 2.3 the number of simple representations of R_v that have a semi-cuspidal decomposition with at least two parts is the number of

ways of writing ν as a sum of multiples of roots α where the arguments $\arg c(\alpha)$ are not all equal. Thus the number of semi-cuspidal simple representations of R_ν is the number of ways of writing ν as a sum of multiples of positive roots all of which have the same argument. \square

Corollary 2.8 *If \mathfrak{g} is finite type and c is a generic charge (i.e. a charge such that $\arg c(\alpha) \neq \arg c(\beta)$ for all $\alpha \neq \beta \in \Delta_+$), then there is a unique cuspidal representation \mathcal{L}_α of R_α for each positive root α , and no others.*

Remark 2.9 *The finite-type case of Theorem 2.3 (and thus Corollary 2.8) have been shown independently by McNamara [McN, 3.1]; this has been extended to affine type by Kleshchev in [Kle].*

Proof of Corollary 2.8. By Corollary 2.7 the only ν for which there is a semi-cuspidal representation are $\nu = k\alpha$ for some $k \geq 1$ and $\alpha \in \Delta_+$, and in all these cases there is only one isomorphism class of semi-cuspidal representation. The semi-cuspidal representation L_α of dimension α must in fact be cuspidal, since there is no element of the root lattice on the line from 0 to α . \square

For minimal roots (i.e. roots α such that $x\alpha$ is not a root for any $0 < x < 1$; see section 1.2), the same arguments used in the proof of Corollary 2.8 shows that the root multiplicity coincides with the number of cuspidal representations. However, this is not always the case for example, Section 3.5 gives an example where this is false for $\widehat{\mathfrak{sl}}_2$ with $\nu = 2\delta$.

We also have the following generalized notion of cuspidal representation where we allow any convex order on Δ_+^{\min} , not just those coming from charges.

Definition 2.10 *Fix a pair $(\alpha, <)$ of a minimal positive root and a convex order on Δ_+^{\min} . We call a simple representation L of R_α **$<$ -(semi)-cuspidal** if L is c -(semi)-cuspidal for some $(\alpha, <)$ -compatible charge c (see Definition 1.8).*

Proposition 2.11 *A simple representation L of R_α for some positive root α is $<$ -(semi)-cuspidal if and only if L is c -(semi)-cuspidal for all $(\alpha, <)$ -compatible charges c .*

Proof. Assume that L is $<$ -cuspidal, and let c be the $(\alpha, <)$ compatible charge from Definition 2.10. Let c' be another $(\alpha, <)$ -compatible charge, and assume L is not cuspidal for c' . Thus, there exists β with $\beta >_{c'} \alpha$ such that $\text{Res}_{\beta, \alpha - \beta}^\alpha L \neq 0$.

Since c' is $(\alpha, <)$ -compatible this implies that $\beta > \alpha$, which, since c is also $(\alpha, <)$ compatible, implies $\beta >_c \alpha$ as well. But L is cuspidal, so $\text{Res}_{\beta, \alpha - \beta}^\alpha L \neq 0$ is a contradiction. Thus L is in fact cuspidal for c' as well. The same argument carries through for semi-cuspidality. \square

Corollary 2.12 *If \mathfrak{g} is finite type then, for any convex order $<$ on Δ_+^{\min} , there is a unique $<$ -cuspidal representation \mathcal{L}_α of R_α for each positive root α , and no others.*

Proof. This follows immediately from Corollary 2.8 using some $(\alpha, <)$ -compatible charge c . \square

2.2. Lusztig data in \mathcal{KR} . Fix a convex order $<$. The following theorem shows that the elements $L \in \mathcal{KR}$ are parameterized by certain lists of semi-cuspidal representations. This gives a generalization of the notion of Lusztig data commonly used in finite type.

Theorem 2.13 *There is a bijection sending a list L_1, \dots, L_h of $<$ -semi-cuspidal representations with $\text{wt}(L_1) > \dots > \text{wt}(L_h)$, to a simple R -module $A(L_1, \dots, L_h)$ uniquely specified by the property that, for every $1 \leq k \leq h-1$, if $L' = A(L_1, \dots, L_k)$ and $L'' = A(L_{k+1}, \dots, L_h)$ then L is the unique simple quotient of $L' \circ L''$.*

In particular, $A(-)$ can be defined inductively by letting $A(L_1, \dots, L_h)$ be the cosocle of $A(L_1, \dots, L_{h-1}) \circ L_h$.

Remark 2.14 *The module $A(L_1, \dots, L_h)$, defined recursively by letting $A(L_1, \dots, L_h)$ be the cosocle of $A(L_1, \dots, L_{h-1}) \circ L_h$, makes sense for any list of modules L_k . However unless one is in the specific situation of Theorem 2.13, there is no guarantee that this algorithm will output a simple, rather than semi-simple, module. We will only use this notation in cases where we know the resulting module to be simple.*

Proof of Theorem 2.13. We proceed by induction on h , the case $h = 1$ being trivial. Fix $1 \leq k \leq h-1$. The simples L' and L'' exist and have the desired properties by induction. Fix a $(\text{wt}(L_k), <)$ -compatible charge c .

For each $j < k$, choose a $(\text{wt}(L_j), <)$ -compatible charge c_j . Since L_j is c_j -semi-cuspidal, no word appearing in the character of L_j has a right factor of weight $\beta \leq \text{wt}(L_j)$. In particular, no word appearing in the character of L_j has a right factor of weight $\beta \leq \text{wt}(L_k)$. This implies that the semi-cuspidal decomposition L'_1, \dots, L'_m of L' for the charge c only involves semi-cuspidals of weight $> \text{wt}(L_k)$. It is also clear that $L'_m = L_k$. Similarly let L''_1, \dots, L''_ℓ be the semi-cuspidal decomposition of L'' . By transitivity $\text{wt}(L'_j) > \beta > \text{wt}(L''_b)$ for all j, b , so we also have $\text{wt}(L'_j) >_c \beta >_c \text{wt}(L''_b)$. Thus

$$\text{wt}(L'_1) >_c \dots >_c \text{wt}(L'_m) >_c \text{wt}(L''_1) >_c \dots \text{wt}(L''_\ell),$$

so $L'_1 \circ \dots \circ L'_m \circ L''_1 \circ \dots \circ L''_\ell$ has a unique irreducible quotient. Since $L' \circ L''$ is a quotient of this, it has a unique irreducible quotient as well.

Now assume $k \geq 2$, and let $M' = A(L_1, \dots, L_{k-1})$ and $M'' = A(L_k, \dots, L_h)$. Since $L'_m = L_k$, the semi-cuspidals for M' for c are just L'_1, \dots, L'_{m-1} . Thus, $M' \circ L_k \circ L''$ is a quotient of $L'_1 \circ \dots \circ L'_{m-1} \circ L'_m \circ L''_1 \circ \dots \circ L''_\ell$ and has unique simple quotient. Thus, $M' \circ M''$ and $L' \circ L''$ have the same unique simple quotient. Iterating this argument shows that this simple is independent of k .

It remains to show that the resulting map from a list L_* of $<$ -semi-cuspidal representations to a simple module is bijective. So, assume that for two different lists L_* and M_* , we arrive at the same simple module L . Let k be minimal such that $L_k \not\cong M_k$. Without loss of generality, we can assume $\text{wt}(L_k) \geq \text{wt}(M_k)$. Let c be a $(\text{wt}(L_k), <)$ -compatible charge. Then the construction above using the charge c and the two lists L_* and M_* expresses L as the quotient of the composition of two different $<_c$ ordered lists of c -semi-cuspidals, one of which contains L_k and one of which does not, which is impossible. Thus $L_k \cong M_k$ for all k , so this assignment is injective. By counting arguments this map must also be surjective. \square

This theorem is a generalization of [KR, Theorem 7.2], which gives exactly the same sort of description of all simple modules, but only applies to the convex orders arising from Lyndon words.

2.3. Saito reflections on \mathcal{KR} . Fix a charge c , and assume $\arg \alpha_i$ is minimal amongst positive roots. Define a new charge c^{s_i} by $c^{s_i}(\nu) = c(s_i(\nu))$. This will not always send Δ_+ to the upper half plane, but it will send it to some half plane, and we can then rotate to make that the upper half plane (in the end we only care about the order on roots, so the precise rotation we use does not matter). Then α_i will be maximal amount positive roots for c^{s_i} .

This operation has a crystal counterpart, the Saito reflection from Section 1.1. We now discuss how this crystal-theoretic operation works when the underlying set of $B(-\infty)$ is identified with \mathcal{KR} .

Note that if $\varphi_i^*(L_1) = \varphi_i^*(L_2) = 0$, then any composition factor L of $L_1 \circ L_2$ is also killed by f_i^* by [KL09, 2.18]; thus, we can consider the action of Saito reflections on these simples.

Lemma 2.15 *If (L_1, L_2) are an unmixing pair (see Definition 2.4) such that $\varphi_i^*(L_1) = \varphi_i^*(L_2) = 0$, then $(\tilde{e}_i^*)^n L$ is the unique simple quotient of*

$$L^{(n)} = \begin{cases} (\tilde{e}_i^*)^n L_1 \circ L_2 & n \leq \epsilon_i^*(L_1) \\ (\tilde{e}_i^*)^{\epsilon_i^*(L_1)} L_1 \circ (\tilde{e}_i^*)^{n-\epsilon_i^*(L_1)} L_2 & \epsilon_i^*(L_1) < n \end{cases}$$

Proof. Since there are no words in the character of L_1 or L_2 beginning with i , the triple $(\mathcal{L}_i^n, L_1, L_2)$ is unmixing. By Lemma 2.5, the induction $\mathcal{L}_i^n \circ L_1 \circ L_2$ has a unique simple quotient.

Thus, if we define a surjective map $\mathcal{L}_i \circ L^{(n-1)} \rightarrow L^{(n)}$, this will show by induction that $L^{(n)}$ has unique simple quotient, and that this quotient agrees with $(\tilde{e}_i^*)^n L$.

If $n \leq \epsilon_i^*(L_1)$, then the map is the obvious one. If $n > \epsilon_i^*(L_1)$, then we use the fact that

$$\mathcal{L}_i \circ (\tilde{e}_i^*)^{\epsilon_i^*(L_1)} L_1 \cong (\tilde{e}_i^*)^{\epsilon_i^*(L_1)} L_1 \circ \mathcal{L}_i,$$

so we have that

$$\mathcal{L}_i \circ L^{(n-1)} \cong (\tilde{e}_i^*)^{\epsilon_i^*(L_1)} L_1 \circ \mathcal{L}_i \circ (\tilde{e}_i^*)^{n-1-\epsilon_i^*(L_1)} L_2$$

which has an obvious surjective map to $L^{(n)}$. \square

Lemma 2.16 *If (L_1, L_2) is an unmixing pair such that $\varphi_i^*(L_1) = \varphi_i^*(L_2) = 0$, and $(\sigma_i(L_1), \sigma_i(L_2))$ is also an unmixing pair, then $\sigma_i(L)$ is the unique simple quotient of $\sigma_i(L_1) \circ \sigma_i(L_2)$.*

Proof. Letting L' be the unique simple quotient of $\sigma_i(L_1) \circ \sigma_i(L_2)$, it suffices to show that

$$(\tilde{e}_i^*)^{\epsilon_i^*(L_1)+\epsilon_i^*(L_2)}L = (\tilde{e}_i)^{\epsilon_i(L_1)+\epsilon_i(L_2)}L'.$$

By Lemma 2.15, the former is the unique simple quotient of $(\tilde{e}_i^*)^{\epsilon_i^*(L_1)}L_1 \circ (\tilde{e}_i^*)^{\epsilon_i^*(L_2)}L_2$ and the latter of $\tilde{e}_i^{\epsilon_i(L_1)}\sigma_i L_1 \circ \tilde{e}_i^{\epsilon_i(L_2)}\sigma_i L_2$. Since $(\tilde{e}_i^*)^{\epsilon_i^*(L_j)}L_j \cong \tilde{e}_i^{\epsilon_i(L_j)}\sigma_i L_j$ for $j = 1, 2$, these modules are isomorphic, and we are done. \square

Proposition 2.17 *Assume that L is a module with $\tilde{f}_i^*L = 0$, and let (L_1, \dots, L_h) be the semi-cuspidal decomposition of L for $<$, then the semi-cuspidal decomposition of $\sigma(L)$ for $<^{s_i}$ is $(\sigma_i L_1, \dots, \sigma_i L_h)$. In particular, the operation σ_i defines a bijection between semi-cuspidal modules for $<$ with $\tilde{f}_i^*L = 0$ and semi-cuspidal modules for $<^{s_i}$ with $\tilde{f}_i^*L = 0$, with inverse given by σ_i^* .*

Proof. The proof is by induction, fixing L , and assuming the proposition for any simple which has a smaller maximal height in its semi-cuspidal decomposition, or a smaller number of components.

If $h > 1$, let $L' = A(L_2, \dots, L_h)$; this is an unmixing pair, and $L = A(L_1, L')$. By induction, the modules $\sigma_i L_j$ are semi-cuspidal and $\sigma(L') = A(\sigma_i L_2, \dots, \sigma_i L_h)$. Thus, $\sigma_i(L_1)$ and $\sigma_i(L')$ are again an unmixing pair, and by Lemma 2.16, we have that $\sigma_i(L) = A(\sigma_i(L_1), \sigma_i(L'))$. Thus, we have that $\sigma_i(L) = A(\sigma_i L_1, \dots, \sigma_i L_h)$.

If L is semi-cuspidal, then we need only establish that $\sigma_i L$ is again semi-cuspidal. Since two corresponding sets of semi-cuspidals for $<$ of weight ν and for $<^{s_i}$ of weight $s_i \nu$ have the same number, we need only show that if L is not a semi-cuspidal module of weight ν , then $\sigma_i L$ is not semi-cuspidal. By induction, we have that $\sigma_i L = A(\sigma_i L_1, \dots, \sigma_i L_h)$, so this fact is manifest. \square

Put another way, we have that:

Corollary 2.18 *Assume $<$ is a convex order such that α_i is minimal and that L_1, \dots, L_h are $<$ -semi-cuspidal representations with $\text{wt}(L_1) > \dots > \text{wt}(L_h) > \alpha_i$. Then*

$$\sigma_i A(L_1, \dots, L_h) \cong A(\sigma_i L_1, \dots, \sigma_i L_h).$$

Corollary 2.18 is a very important technical tool for us. In particular it allows us to reduce questions about cuspidal representations to the case where the root is simple, using the following.

Lemma 2.19 *Fix a simple L and convex order $<$, and assume the semi-cuspidal decomposition of L is*

$$L = A(\mathcal{L}_\alpha^{n_\alpha}, \dots, \mathcal{L}_\beta^{n_\beta}).$$

If \mathfrak{g} is finite type or is affine with $\alpha > \delta$, then there is a finite sequence $\sigma_{i_1}, \dots, \sigma_{i_k}$ of Saito reflections such that for all j , $\varphi_j^(\sigma_{i_{j-1}} \cdots \sigma_{i_1} L) = 0$ and $s_{i_k} \cdots s_{i_1} \alpha$ is a simple root α_m . Thus*

$$\sigma_{i_k} \cdots \sigma_{i_1} L = A(\mathcal{L}_{\alpha_m}^{n_\alpha}, \dots, \sigma_{i_k} \cdots \sigma_{i_1} \mathcal{L}_\beta^{n_\beta})$$

If \mathfrak{g} is finite type or affine with $\beta < \delta$, then a similar list of dual Saito reflections $\sigma_{i_1}^, \dots, \sigma_{i_h}^*$ exists and*

$$\sigma_{i_h}^* \cdots \sigma_{i_1}^* L = A(\sigma_{i_k}^* \cdots \sigma_{i_1}^* \mathcal{L}_\alpha^{n_\alpha}, \dots, \mathcal{L}_{\alpha_\ell}^{n_\beta}).$$

Proof. The two statements are swapped by the Kashiwara involution, so we need only prove the first. We proceed by induction on the number of positive roots $\eta > \alpha$ (which is thus finite). The case when α is maximal with respect to $>$ (and hence is simple) is obvious; so assume that for some $k \geq 1$ the statement is known for all charges c and all positive roots α with at most $k - 1$ positive roots $\eta > \alpha$. Fix c and α with exactly k roots $> \alpha$. Let α_{i_1} be the maximal root (which is necessarily simple). Then $\varphi_{i_1}^*(L) = 0$, since $\mathcal{L}_{\alpha_{i_1}}$ does not appear in its cuspidal decomposition. Furthermore, the positive roots $\beta >_{s_i} s_{i_1} \alpha$ are exactly those of the form $\beta = s_i \beta'$ for $\beta' > \alpha$ and $\beta' \neq \alpha_{i_1}$. Thus, the number is one less than for α and $>$, and we may now apply the inductive hypothesis to $\sigma_{i_1} L$. Thus, there exist i_2, \dots, i_k such that $\sigma_{i_k} \cdots \sigma_{i_1} \mathcal{L}_\alpha = \mathcal{L}_{\alpha_m}$ and $\varphi_j^*(\sigma_{i_{j-1}} \cdots \sigma_{i_1} L) = 0$ for all j . This completes the proof. \square

Definition 2.20 *Fix a convex order $>$. For each $b \in B(-\infty)$ and each real root α such that $\alpha > \beta$ for all imaginary roots β , we define an integer $\mathfrak{a}_\alpha(b)$. This is done inductively by setting $\mathfrak{a}_{\alpha_i}(b) = \varphi_i(b)$ if α_i maximal in the order $>$, and*

$$\mathfrak{a}_\alpha(b) = \mathfrak{a}_{s_i \alpha}^{s_i}(\sigma_i^*(f_i^{\varphi_i(b)} b))$$

*for all other roots, where $\mathfrak{a}_{s_i \alpha}^{s_i}$ denotes Lusztig data calculated for the order $>$ twisted by the reflection s_i . The collection of such $\mathfrak{a}_\alpha(b)$ is called the **crystal-theoretic real Lusztig data** for b with respect to $>$.*

Similarly, define dual Lusztig data \mathfrak{a}_α^ for all real roots smaller than all imaginaries by switching starred and unstarred operators. For finite type groups, this is just the Lusztig data for the opposite order, but in infinite types is new information.*

Remark 2.21 *As was recently explained by Kato [Kat], in symmetric type there are in fact equivalences of categories*

$$(4) \quad \left\{ L : \mathcal{F}_i(L) = 0 \right\} \leftrightarrow \left\{ L : \mathcal{F}_i^*(L) = 0 \right\}$$

which induce Saito reflections on the set of simples. The proof in [Kat] uses the geometry of quiver varieties, which is only available in symmetric type; it seems likely, however, that there is an algebraic version of Kato's functor as well.

3. KLR POLYTOPES AND MV POLYTOPES

Having developed this combinatorics for understanding representations of KLR algebras, we now turn to encoding this information in polytopes.

3.1. KLR polytopes.

Definition 3.1 For each simple L , the **character polytope** P_L is the convex hull of the weights ν' such that $\text{Res}_{\nu', \nu-\nu'}^\nu L \neq 0$. Equivalently, this is the convex hull of the elements $\alpha_{i_1 \dots i_\ell}$ where $\dim(e_i L) \neq 0$.

Geometrically, we can think of every word \mathbf{i} appearing in $\text{ch}(L)$ as a path in \mathfrak{h}^* ; the polytope P_L can also be described as the convex hull of all these paths.

These polytopes live in the dual of the Cartan \mathfrak{h}^* , which has a natural height function ρ^\vee . This orients each edge of the polytope, and gives every face F a maximal vertex v_t and minimal vertex v_b . We associate a KLR algebra R_F to each face F by $R_F := R_{v_t-v_b}$. Thus we have the subalgebra of R_ν given by $R_{v_b} \otimes R_F \otimes R_{\nu-v_t}$, and can consider the restriction functor Res_F^ν restricting to this subalgebra.

Proposition 3.2 For any simple L and face F of P_L , the restriction $\text{Res}_F^\nu L$ is simple and thus the outer tensor of three simples $L' \boxtimes L_F \boxtimes L''$.

Proof. Choose a function ϕ that obtains its minimum on P_L exactly on F , and consider the charge $\rho^\vee + i\phi$; the simple L is the unique simple quotient of an increasing induction of semi-cuspidals $L_1 \circ \dots \circ L_h$; let k be smallest index where $\phi(\text{wt}(L_k)) = 0$ and m the largest such index. Let L' be the simple quotient of $L_1 \circ \dots \circ L_{k-1}$, let L_f be the simple quotient of $L_k \circ \dots \circ L_m$, and let L'' be the simple quotient of $L_{m+1} \circ \dots \circ L_h$.

Thus, $L' \circ L_f \circ L''$ is a quotient of $L_1 \circ \dots \circ L_h$, and thus has a unique simple quotient, which is L . On the other hand, $\text{Res}_F^\nu(L' \circ L_f \circ L'') = L' \boxtimes L_F \boxtimes L''$, so L must also restrict to this same module. \square

Definition 3.3 Fix $L \in \mathcal{KLR}$. The **KLR polytope** \tilde{P}_L of L is the polytope P_L along with the data of the isomorphism class of the semi-cuspidal representation L_E associated to each edge E of P_L as above. We denote by $\mathbf{P}^{\mathcal{KLR}}$ the set of all KLR polytopes.

Remark 3.4 The representations which can appear as labels in \tilde{P}_L are not arbitrary; they must be semi-cuspidal for any charge which includes that edge in its walk.

Proposition 3.5 Every edge of P_L is parallel to a positive root of \mathfrak{g} . That is, P_L is a pseudo-Weyl polytope.

Proof. For any edge E , we can pick a generic function φ which achieves its maximum on P_L exactly on E . Since at most one element of Δ_+^{\min} is parallel to E , a generic such φ produces a charge generic in the usual sense: it induces a total order on the words appearing in simple representations of R_E . Furthermore, L_E is semi-cuspidal for this charge so by Corollary 2.7 is a multiple of a positive root. \square

Remark 3.6 *In finite type, there is exactly one semi-cuspidal simple representation of $R_{k\alpha}$ for each positive root α and $k \geq 1$, so the decoration is superfluous. The KLR polytope \tilde{P}_L is completely determined by the character polytope P_L , and $\mathbf{P}^{\mathcal{KLR}}$ can be thought of as simply a set of pseudo-Weyl polytopes.*

We denote the path through P determined by a convex order $<$ in Lemma 1.12 by $P^<$. We obtain a list simple modules of simple modules L_1, \dots, L_h with $\text{wt}(L_1) > \dots > \text{wt}(L_h)$ by taking the modules corresponding to the edges in $P^<$.

Proposition 3.7 $L = A(L_1, \dots, L_h)$.

Proof. We induct on h . Let E be the last edge in $P^<$, and consider $\text{Res}_E^\vee L$; this is of the form $L' \boxtimes L_h$. Obviously the edges in $P_{L'}$ and P_L coincide along the walk corresponding to $<$ up to but not including E . Thus, L_1, \dots, L_{h-1} are the simples associated to this walk for L' by the algorithm above, and by the inductive assumption, $L' = A(L_1, \dots, L_{h-1})$. Thus, $A(L_1, \dots, L_h)$ is the unique simple quotient of $L' \circ L_h$. Of course, L is also a quotient of this module by Frobenius reciprocity, so these simples coincide. \square

Proposition 3.7 has the following immediate consequences:

Corollary 3.8 *The polytope P_L with the labeling of just its edges along $P^<$ uniquely determines the simple L . In particular, the map $L \mapsto \tilde{P}_L$ defines a bijection $\mathcal{KLR} \rightarrow \mathbf{P}^{\mathcal{KLR}}$.* \square

Corollary 3.9 *The function sending a labelled polytope to the list of semi-cuspidal representations attached to $P^<$ is a bijection from $\mathbf{P}^{\mathcal{KLR}}$ to the set of ordered lists of semi-cuspidal representations, for any convex order.* \square

This still leaves us with the question of a combinatorial description of the set $\mathbf{P}^{\mathcal{KLR}}$. As we shall see in finite type, this set is precisely the MV polytopes. An answer in general could serve as a satisfying definition of general-type Mirković-Vilonen polytopes, and would shed significant light on the structure of representations of KLR algebras.

Since the map which takes L to \tilde{P}_L is injective, the crystal structure on \mathcal{KLR} gives rise to a crystal structure on $\mathbf{P}^{\mathcal{KLR}}$. Using Corollary 3.8, we can now give a combinatorial description of the resulting crystal operators.

Proposition 3.10 *To apply the operator \tilde{f}_i to $\tilde{P} \in \mathbf{P}^{\text{KLR}}$, we choose a convex order with α_i maximal, and read the path determined by that order to obtain a list of semi-cuspidal representations L_1, \dots, L_h corresponding to increasing roots in that order. If $L_h = \mathcal{L}_i^k$ for some $k \geq 1$, then*

$$\tilde{f}_i \tilde{P} = \tilde{P}_{A(L_1, \dots, L_{h-1}, \mathcal{L}_i^{k-1})}.$$

If $L_h \not\cong \mathcal{L}_i^k$, then $\tilde{f}_i \tilde{P} = 0$.

Proof. If $L_h \not\cong \mathcal{L}_i^k$, then $A(L_1, \dots, L_h)$ is a quotient of $L_1 \circ \dots \circ L_h$, whose character is a quantum shuffle of words not ending in i , and thus only contains words not ending in i . Thus, $\tilde{f}_i A(L_1, \dots, L_h) = 0$ follows immediately. On the other hand $A(L_1, \dots, L_{h-1}, \mathcal{L}_i^k)$ is a quotient of

$$A(L_1, \dots, L_{h-1}) \circ \mathcal{L}_i^k \cong (A(L_1, \dots, L_{h-1}) \circ \mathcal{L}_i^{k-1}) \circ L_i \rightarrow A(L_1, \dots, L_{h-1}, \mathcal{L}_i^{k-1}) \circ \mathcal{L}_i,$$

and thus by definition is $\tilde{e}_i A(L_1, \dots, L_{h-1}, \mathcal{L}_i^{k-1}) = A(L_1, \dots, L_h)$ and we are done. \square

We also have the following, which is simply a restatement of Corollary 2.18 in the language of polytopes.

Proposition 3.11 *To apply a Saito reflection functor σ_i to a polytope $\tilde{P} \in \mathbf{P}$ with $\tilde{f}_i \tilde{P} = 0$, choose a convex order and let L_1, \dots, L_h be as before. Then*

$$\sigma_i \tilde{P} = \tilde{P}_{A(\sigma_i L_1, \dots, \sigma_i L_h)}.$$

Comparing Corollary 2.18 with Propositions 3.10 and 3.11 immediately yields that:

Corollary 3.12 *For any simple L with corresponding element b in $B(-\infty)$ and any convex order $>$, the polytope Lusztig data $a_\alpha(L)$ agrees with the crystal-theoretic Lusztig data $a_\alpha(b)$ for all real roots either greater than any imaginary root, and with the dual data $a_\alpha^*(b)$ for any root lesser than any imaginary root.*

In particular, Corollary 3.12 allows us to complete the proof of Theorem A.

Proof of Theorem A. Two pseudo-Weyl polytopes for a finite dimensional Lie coincide if and only if their Lusztig data are identical for every convex order. The Lusztig data of the MV polytope P_b is given by the crystal-theoretic Lusztig data $a_*(b)$ and that of the KLR polytope P_L is given by $a_*(L)$ by definition. Thus, Corollary 3.12 shows that these polytopes coincide. \square

3.2. The crystal corresponding to a face. Fix a charge c . The set of real roots with argument $\pi/2$ are the real roots of some root system Δ_c . Let \mathfrak{g}_c be the corresponding Lie algebra. Let β_1, \dots, β_s be the simple roots of \mathfrak{g}_c . To avoid the possibility of confusion between the indexing set of the roots of \mathfrak{g} and those of \mathfrak{g}_c , we will always index the latter with underlined numbers.

Lemma 3.13 *Let β be a simple root for \mathfrak{g}_c . Then there is a finite word $\mathbf{i} = i_k \cdots i_1$ such that $s_{i_k} \cdots s_{i_1} \beta$ is a simple root α_i for \mathfrak{g} and, for each $1 \leq j \leq k$, $\arg \alpha_{i_j} \neq \pi/2$ with respect to the charge $s_{i_{j-1}} \cdots s_{i_1} c$.*

Proof. Recall that the height $\text{ht } \alpha$ of a root α is the sum of the coefficients when that root is expressed in terms of simple roots. Given any non-simple positive root α , one can always find i such that $\text{ht}(s_i \alpha) < \text{ht}(\alpha)$. Since height is finite, we can find an expression

$$\alpha_i = s_{i_k} \cdots s_{i_1} \beta$$

such that, for each j , $\text{ht}(s_{i_j} \cdots s_{i_1} \beta) < \text{ht}(s_{i_{j-1}} \cdots s_{i_1} \beta)$. Now, for each j , $s_{i_{j-1}} \cdots s_{i_1} \beta$ is a simple root for $\mathfrak{g}_{s_{i_{j-1}} \cdots s_{i_1} c}$, and if $\arg \alpha_{i_j} = \pi/2$ then this must be a simple root for $\mathfrak{g}_{s_{i_{j-1}} \cdots s_{i_1} c}$ as well. But then $s_{i_j} s_{i_{j-1}} \cdots s_{i_1} \beta$ would have to be of the form $s_{i_{j-1}} \cdots s_{i_1} \beta + k\alpha_i$ for some $k \geq 0$, and this contradicts the fact that $\text{ht}(s_{i_j} \cdots s_{i_1} \beta) < \text{ht}(s_{i_{j-1}} \cdots s_{i_1} \beta)$. Thus $\arg \alpha_{i_j} \neq \pi/2$ with respect to the charge $s_{i_{j-1}} \cdots s_{i_1} c$, as required. \square

Proposition 3.14 *The operators given by*

$$\begin{aligned} \tilde{\mathbf{e}}_{\underline{i}} L &= \text{cosoc}(L \circ \mathcal{L}_{\beta_i}), & \tilde{\mathbf{e}}_{\underline{i}}^* L &= \text{cosoc}(\mathcal{L}_{\beta_i} \circ L), \\ \tilde{\mathbf{f}}_{\underline{i}} L &= \text{cosoc}(\text{Hom}_{R_{v-\beta_i} \otimes R_{\beta_i}}(R \circ \mathcal{L}_{\beta_i}, L)), & \tilde{\mathbf{f}}_{\underline{i}}^* L &= \text{cosoc}(\text{Hom}_{R_{\beta_i} \otimes R_{v-\beta_i}}(\mathcal{L}_{\beta_i} \circ R, L)) \end{aligned}$$

define a \mathfrak{g}_c combinatorial bicrystal structure on the set of all c -semi-cuspidal representations of argument $\pi/2$. The additional combinatorial data is given by

$$\begin{aligned} \underline{\text{wt}}(L) &= \text{wt}(L)|_{\mathfrak{h}_c} \\ \varphi_{\underline{i}}(L) &= \max\{n \mid \tilde{\mathbf{f}}_{\underline{i}}^n(L) \neq 0\}, & \varphi_{\underline{i}}^*(L) &= \max\{n \mid (\tilde{\mathbf{f}}_{\underline{i}}^*)^n(L) \neq 0\} \\ \epsilon_{\underline{i}}(L) &= \varphi_{\beta_i}(L) - \alpha_{\beta_i}^\vee(\tilde{\text{wt}}(L)), & \epsilon_{\underline{i}}^*(L) &= \varphi_{\beta_i}^*(L) - \alpha_{\beta_i}^\vee(\tilde{\text{wt}}(L)). \end{aligned}$$

Proof. First we must show that these operators are well defined. If α_i has argument $> \pi/2$ we can apply Saito reflection σ_i and if α_i has argument $< \pi/2$, we can apply σ_i^* , and in either case it follows that checking $\tilde{\mathbf{e}}_{\beta_k}$ is well defined on c -semi-cuspidal representations of argument $\pi/2$ is equivalent to checking that $\tilde{\mathbf{e}}_{s_i \beta_k}$ is well defined on $s_i c$ -semi-cuspidal representations of argument $\pi/2$. Thus Lemma 3.13 allows us to reduce to the case when $\beta_k = \alpha_i$ is simple. But in this case $\tilde{\mathbf{e}}_{\beta_k}$ is just the ordinary crystal operator f_k for \mathfrak{g} , which we know is well defined.

That these operators form a combinatorial bicrystal is then immediate from the fact that the whole crystal structure is a combinatorial bicrystal. \square

We say that an element of a bicrystal is **lowest weight** if it is killed by all lowering Kashiwara operators, both starred and unstarred.

Lemma 3.15 *Assume L^h is a c -semi-cuspidal representation of argument $\pi/2$ which is lowest weight for the bicrystal structure with $\text{wt}(L^h)$ a null vector for the root system*

Δ_c . Then the component generated by L^h under the crystal operators $\tilde{e}_{\underline{j}}$ is the same as the component generated by L^h under the $\tilde{e}_{\underline{j}}^*$.

Proof. The proof is by induction on the sum $d(L)$ of the coefficients of the expression for $\text{wt}(L) - \text{wt}(L^h)$ in terms of the β_k .

If $d(L) = 1$ then $\text{wt}(L) - \text{wt}(L^h) = \beta_{\underline{j}}$, and $L = \tilde{f}_{\underline{j}} L^h$ for some j . As in the proof of Proposition 3.14 we can apply Saito reflections until $\beta_k = \alpha_i$. In that case it follows from Proposition 1.4 and the fact that the whole crystal is $B^{\mathfrak{g}}(-\infty)$ that $\tilde{f}_{\underline{j}} L^h = \tilde{f}_{\underline{j}}^* L^h$, so the claim holds.

Now assume the result holds for all L' with depth $d(L') < d$. The proof is symmetric in the two structures, so it suffices to fix $L = \tilde{e}_{\underline{j}_d} \tilde{e}_{\underline{j}_{d-1}} \cdots \tilde{e}_{\underline{j}_1} L^h$ and show that it lies in the starred component of L^h . By the $d = 1$ case this is isomorphic to $L = \tilde{e}_{\underline{j}_d} \tilde{e}_{\underline{j}_{d-1}} \cdots \tilde{e}_{\underline{j}_1}^* L^h$. From here it is clear that $L|_{R(\text{wt}(L) - \beta_{\underline{j}_1}) \otimes R(\beta_{\underline{j}_1})} \neq 0$, so $\tilde{f}_{\underline{j}_1} L \neq 0$. Since we know that \mathcal{KR} is isomorphic to $B(-\infty)$ as a bi-crystal, by Proposition 1.4 we must be in one of the following two cases:

(1) If $\underline{j}_1 \neq \underline{j}_d$ or $\underline{j}_1 = \underline{j}_d$ and $\tilde{f}_{\underline{j}_1}^* \tilde{f}_{\underline{j}_1} L = \tilde{f}_{\underline{j}_1} \tilde{f}_{\underline{j}_1}^* L$ then $L = \tilde{e}_{\underline{j}_d} \tilde{e}_{\underline{j}_{d-1}} \tilde{f}_{\underline{j}_1}^* \tilde{f}_{\underline{j}_1} L$. The module $\tilde{f}_{\underline{j}_1} L$ is manifestly in the component of the unstarred component of L^h , and thus by induction in the starred component as well. Using the inductive hypothesis again, $\tilde{e}_{\underline{j}_d} \tilde{f}_{\underline{j}_1}^* \tilde{f}_{\underline{j}_1} L$ is in the starred components of L^h , and so L is as well.

(2) If $\underline{j}_1 = \underline{j}_d$ and $\tilde{f}_{\underline{j}_1} L = \tilde{f}_{\underline{j}_1}^* L$, then $\tilde{f}_{\underline{j}_1} L = \tilde{f}_{\underline{j}_1}^* L$ is in both the starred and unstarred component of L^h . Since $L = \tilde{e}_{\underline{j}_d}^* \tilde{f}_{\underline{j}_1}^* L$, we see that L is also in the starred component. \square

Proposition 3.16 Assume L^h is a c -semi-cuspidal representations of argument $\pi/2$ which is lowest weight for the bicrystal structure with $\text{wt}(L^h)$ a null vector for the root system Δ_c . Then the component generated by L^h under all $\tilde{e}_{\underline{j}}, \tilde{e}_{\underline{j}}^*$ is isomorphic (as a bicrystal) to the infinity crystal $B^{\mathfrak{g}}(-\infty)$.

Proof. By Lemma 3.15, it suffices to check conditions (i)-(v) of Corollary 1.4. To check condition (i), consider the module $\mathcal{L}_{\beta_{\underline{i}}} \circ L \circ \mathcal{L}_{\beta_{\underline{i}}}$; both $\tilde{e}_{\underline{i}} \tilde{e}_{\underline{i}}^* L$ and $\tilde{e}_{\underline{i}}^* \tilde{e}_{\underline{i}} L$ are quotients of this module. Since $\beta_{\underline{j}}$ and $\beta_{\underline{i}}$ are simple among the roots with c -argument $\pi/2$, there is a deformation c' of c such that $\beta_{\underline{i}}$ is minimal among the roots with $c(\beta)$ imaginary, and $\beta_{\underline{j}}$ is maximal, and $c'(P_L)$ is in the region of the half-plane with argument $c'(\beta_{\underline{j}})$. Now, let (L_1, \dots, L_h) be the semi-cuspidal decomposition of L . By assumption, each L_i is semi-cuspidal for c , so their weights have argument between that of $c'(\beta_{\underline{j}})$ and $c'(\beta_{\underline{i}})$ (inclusive). Thus, $\mathcal{L}_{\beta_{\underline{i}}} \circ L_1 \circ \cdots \circ L_h \circ \mathcal{L}_{\beta_{\underline{i}}}$ has a unique simple quotient by Theorem 2.3. Since $\mathcal{L}_{\beta_{\underline{i}}} \circ L \circ \mathcal{L}_{\beta_{\underline{i}}}$ is a quotient of this module, this uniqueness proves $\tilde{e}_{\underline{i}} \tilde{e}_{\underline{i}}^* L = \tilde{e}_{\underline{i}}^* \tilde{e}_{\underline{i}} L$.

Each of the conditions (ii)-(v) only involves a single simple root $\beta_{\underline{i}}$. Furthermore, each of these conditions holds before Saito reflection if and only if it holds after by

Corollary 2.18. Thus, as in the proof that operators are well-defined (Proposition 3.14), we can apply Saito reflections until β_i is simple, in which case these conditions follow from the isomorphism of \mathcal{KR} with $B(-\infty)$ for \mathfrak{g} . \square

Corollary 3.17 *If \mathfrak{g}_c is of finite type, then the set of all c -semi-cuspidal representations of argument $\pi/2$ along with the operators from Proposition 3.14 is isomorphic to $B^{\mathfrak{g}_c}(-\infty)$ as a bicrystal.*

Proof. The trivial representation satisfies all the condition of Proposition 3.14, so generates a copy of $B^{\mathfrak{g}_c}(-\infty)$. Since all roots of \mathfrak{g}_c are real, the number of these representations of a given weight is exactly the Kostant partition function of \mathfrak{g}_c , so this exhausts the set of c -semi-cuspidal representations of argument $\pi/2$. \square

Corollary 3.18 *If both \mathfrak{g} and \mathfrak{g}_c are of affine type, then the set of all c -semi-cuspidal representations of argument $\pi/2$ is isomorphic to a direct sum of copies of $B^{\mathfrak{g}_c}(-\infty)$, and all lowest weight elements L^h have $\text{wt}(L^h) = k\delta$ for some k , and the number of lowest weight elements of weight $k\delta$ is the number of q -multipartitions of k , where $q = r - s = \text{rk } \mathfrak{g} - \text{rk } \mathfrak{g}_c$.*

Proof. By Proposition 3.14 it suffices to show that all lowest weight elements for the unstarred \mathfrak{g}_c crystal structure are also lowest weight for the starred \mathfrak{g}_c crystal structure, that they must have weight $k\delta$, and that there are the right number for each k . We proceed by induction on weight of (potential) lowest weight elements. In weight 0, there is one lowest weight element, and it satisfies all the conditions. So, assume these conditions hold for all lowest weight elements of depth $\leq k\delta$ for some $k \geq 0$. By Proposition 3.14, each of these lowest weight elements generate a copy of $B^{\mathfrak{g}_c}(-\infty)$. The generating function for the number of c -stable reps of argument $\pi/2$ is exactly

$$a(t) = \prod_{\alpha \in \Delta_c} \frac{1}{(1 - t^\alpha)^{\dim \mathfrak{g}_\alpha}}.$$

By comparing with the Kostant partition function

$$b(t) = \prod_{\alpha \in \Delta_c} \frac{1}{(1 - t^\alpha)^{\dim(\mathfrak{g}_c)_\alpha}}$$

for \mathfrak{g}_c , we see that

$$\frac{b(t)}{a(t)} = \prod_{k \geq 1} \frac{1}{(1 - t^{k\delta})^q}$$

is just the generating function of the number of q -multipartitions with variable t^δ . We know that at $k\delta$ and below, the number of lowest weight elements of the bicrystal structure exactly matches this partition function. Thus, we see that the copies of $B^{\mathfrak{g}_c}(-\infty)$ exhaust all c -stable elements of depth less than $(k + 1)\delta$, and miss exactly the

number of q -multipartitions of $k+1$ in that depth. These must all be lowest weight for both structures, and satisfy the necessary conditions, so the induction proceeds. \square

Proposition 3.19 *Assume \mathfrak{g} is of affine type. Fix an irreducible M which is lowest weight for the \mathfrak{g}_c crystal structure, and let N be in the component generated by the trivial representation. Then $L = M \circ N = N \circ M$ is irreducible and $N \mapsto M \circ N$ is a bicrystal isomorphism between the component of the trivial representation and that of M .*

Proof. For any list of weights ν_1, \dots, ν_m , let e_{ν_1, \dots, ν_m} be the idempotent that projects to all sequences which consist of a block of strands summing to ν_1 , a block summing to ν_2 , etc.

Choose any infinite list of nodes j_1, j_2, \dots in the Dynkin diagram in which each node appears infinitely many times (for example, we can pick an order and cycle). In the crystal $B^{\text{gc}}(-\infty)$, the module N has a string parameterization $N = \tilde{e}_{j_1}^{a_1} \tilde{e}_{j_2}^{a_2} \cdots \tilde{e}_{j_\ell}^{a_\ell} L_\emptyset$, where a_1 is maximal number of times \tilde{f}_{j_1} can be applied to N before it becomes 0, a_2 the maximal number of times \tilde{f}_{j_2} can be applied to $\tilde{f}_{j_1} N$, etc.

By definition N is a simple quotient of $\mathcal{L}_{j_\ell}^{a_\ell} \circ \mathcal{L}_{j_{\ell-1}}^{a_{\ell-1}} \circ \cdots \circ \mathcal{L}_{j_1}^{a_1}$. By the definition of the string parameterization, this quotient map must kill the image of $e_{\mathbf{a}'} = e_{a'_\ell \beta_{j_\ell}, \dots, a'_1 \beta_{j_1}}$ where $\mathbf{a}' > \mathbf{a}$ in lexicographic order. Let $L_{\mathbf{a}}$ denote the quotient of this induction by the submodule generated by the image of all such $e_{\mathbf{a}'}$.

Each of the roots β_{j_j} is minimal, so $\mathcal{L}_{j_{\ell-1}}$ is necessarily cuspidal for c , not just semi-cuspidal. Thus, the space $e_{\mathbf{a}} L_{\mathbf{a}}$ is spanned by diagrams which permute the simple terms of the induction. But any diagram that gives a non-trivial permutation must factor through the image of an idempotent $e_{\mathbf{a}'}$ which we have killed (compare with the argument in [KL09, 3.7]). Thus, we see that $e_{\mathbf{a}} L_{\mathbf{a}}$ is just a copy of $\mathcal{L}_{j_\ell}^{a_\ell} \boxtimes \cdots \boxtimes \mathcal{L}_{j_1}^{a_1}$, and thus is simple. Now, we can apply the usual argument to show that $L_{\mathbf{a}}$ has unique simple quotient—any proper submodule is killed by $e_{\mathbf{a}}$, so their sum is as well, and thus is still proper.

Since M can have no final word of weight β_j for any β_j , the same argument as above shows that $e_{k\delta, \mathbf{a}}(M \circ L_{\mathbf{a}})$ is simple, and therefore $M \circ L_{\mathbf{a}}$ has a unique simple quotient as well. But $M \circ N$ is a quotient of this module, so it has a unique simple quotient. On the other, it is easy to see by induction that the module $\tilde{e}_{j_1}^{a_1} \tilde{e}_{j_2}^{a_2} \cdots \tilde{e}_{j_\ell}^{a_\ell} M$ is a quotient of this as well, so this is the unique simple quotient of $M \circ N$. In particular, the map $N \rightarrow A(M, N)$ commutes with the ordinary crystal operators.

The character of $M \circ N$ is the shuffle of the characters of M and N . Since $M \boxtimes N$ is killed in no simple quotient, the cosocle (which we know to be simple) contains the pairwise concatenation of the words in these characters in the obvious order. Similarly the concatenation in the opposite order is contained in the socle, since any shuffle can be multiplied by more crossings to pull all strands from M to the far right.

If \mathbf{i} is a non-trivial word in the character of M , then the weight of any prefix \mathbf{i}_p is either less than $<_c \delta$ or is a multiple of δ . In particular, any word in the character of $M \circ N$ which begins with blocks that step along $\beta_{i_1}, \dots, \beta_{i_d}$ for an arbitrary sequence i_1, \dots, i_d , where $d = \sum a_j$ is the depth of N , must lie in the socle, since those steps can only be obtained by pulling all the strands from N to the far left.

On the other hand, the cosocle is obtained from M by applying d crystal operators to N , so it can also be obtained by applying d starred crystal operations. Thus, there is at least one word in the character of the cosocle which begins with $d = \sum a_i$ many steps along β_i for some i . It follows that the natural map from the socle to the cosocle must be non-zero, thus surjective. Thus the socle is not contained in the radical so, since $M \circ N$ has a unique simple quotient, it must be that $M \circ N$ is in fact simple.

Notice also that the natural map from $N \circ M$ to the socle of $M \circ N$ must be non-zero and thus an isomorphism. Hence $N \circ M \simeq M \circ N$.

We have already established that $N \rightarrow A(M, N) = M \circ N$ is a crystal isomorphism for the unstarred operators; the symmetric argument for $N \circ M$ establishes that it is for the starred operators as well. \square

Lemma 3.20 *Fix a charge c and assume α_i has argument $< \pi/2$ for c . Then Saito reflection σ_i induces a crystal isomorphism between the crystals associated to c and $s_i \cdot c$. Similarly, if α_i has argument $> \pi/2$ for c , then Saito reflection σ_i^* induces a crystal isomorphism between the crystal for $\arg c = \pi/2$ and that for $\arg s_i \cdot c = \pi/2$.*

Proof. This is immediate from the definition of the crystal structure for the face and Corollary 2.18. \square

3.3. Finite type polytopes.

Lemma 3.21 *Fix a charge c such that \mathfrak{g}_c is rank 2 and finite type. Fix $L \in \mathcal{KLR}$ which is cuspidal of argument $\pi/2$ for c , and recall from Corollary 3.17 that L corresponds to a vertex in $B^{\mathfrak{g}_c}(-\infty)$. Then the face of P_L defined by c agrees with the usual MV polytope corresponding to $L \in B^{\mathfrak{g}_c}(-\infty)$.*

Proof. In rank 2 there are exactly two convex orders. We fix the order of the form

$$(5) \quad \eta_1 < \eta_2 < \dots < \eta_{N-1} < \eta_N,$$

and recall that η_1, η_N are simple for \mathfrak{g}_c . The Lusztig data for L with respect to a deformation of c will only involve these roots, and they will come in one of the two possible convex orders. Let $(a_1(P_L), \dots, a_n(P_L))$ be the Lusztig data with respect to order (5), and $(\bar{a}_1(P_L), \dots, \bar{a}_n(P_L))$ be the Lusztig data with respect to the opposite order.

We must check the conditions of Proposition 1.16. Condition (i) is immediate from the definitions of KLR polytopes. Condition (ii) is a special case of Proposition 3.10.

To see condition (iii), notice that we can reflect η_1 to a simple root for \mathfrak{g} using a finite number of reflections, and these can be chosen so that they always satisfy one of the two conditions of Proposition 3.11 (since the only way a reflection can fail to satisfy one of these conditions is if η_N is simple, but then reflection in η_N actually increases the length of η_1). The statement then follows from Proposition 3.11. \square

Corollary 3.22 *Fix a charge c such that all roots with argument $\pi/2$ are real. For each $L \in \mathcal{KLR}$, the face F of P_L defined by c is an MV polytope for \mathfrak{g}_c . In particular, if \mathfrak{g} is of finite type, then P_L agrees with the usual MV polytope.*

Proof. For any 2-face of F , we can choose a charge c such that this 2-face is the locus where c obtains its maximum on P_L . By Lemma 3.21 this 2-face is a rank 2 MV polytope. Thus by Kamnitzer’s characterization of MV polytopes in terms of 2-faces (i.e. Theorem 1.17), F is an MV polytope. \square

3.4. Affine polytopes. Outside of finite type, the conventional definition of MV polytope fails, although as shown in [BKT], an alternate geometric definition can be extended to symmetric affine type. We propose to use the decorated polytope \tilde{P}_L as the “general type MV polytopes.” This construction is not completely combinatorial, as the decoration consists of various representations of KLR algebras. However, in affine type, we can extract purely combinatorial objects.

For the rest of this section fix \mathfrak{g} of affine type with rank $r + 1$. As usual, we label the simple roots of \mathfrak{g} by $\alpha_0, \dots, \alpha_r$ with α_0 being the distinguished vertex as in [Kac90]. We will first prove some technical results concerning the structure of the semi-cuspidal representations of KLR algebras whose weight is a multiple of δ . These will allow us to precisely define the partitions π^\vee associated with a simple L in the introduction, and will be used in the next section to prove Theorem B.

Consider the projection $p: \alpha_i \rightarrow \bar{\alpha}_i$ for $i \neq 0$, $\delta \rightarrow 0$ from affine root space to the root space for $\mathfrak{g}_{\text{fin}}$, the Lie algebra attached to the Dynkin diagram with the 0 node removed, where we use $\bar{\alpha}$ to denote roots in the finite type root system. In all cases other than $A_{2n}^{(2)}$ the image of this map is exactly the set of finite type roots along with 0 (this can be seen by checking that p sends the simple affine roots to a set of finite type roots including all the simples, and using the affine Weyl group). For $A_{2n}^{(2)}$, the image also contains $\alpha/2$ for each of the long roots α in the finite type root system.

For each chamber coweight $\gamma = \theta\omega_i^\vee$ in the finite type root system (i.e. each element in the Weyl group orbit of a fundamental coweight), define a charge c_γ by

$$c_\gamma(\alpha) = \langle \gamma, p(\alpha) \rangle + i\rho^\vee(\alpha).$$

The set of roots with argument $\pi/2$ for c_γ is a rank r affine sub-root system. Hence for any L , c_γ defines a vertical face of P_L , and this is generically dimension r .

Let $\mathfrak{g}_{\text{fin},\gamma}$ be the semi-simple part of the subalgebra of $\mathfrak{g}_{\text{fin}}$ given by the sum of weight spaces on which γ vanishes. For each choice of basis $\eta_1, \dots, \eta_{r-1}$ in the root

system of $\mathfrak{g}_{\text{fin}, \gamma}$ there is a unique $\mathfrak{g}_{\text{fin}}$ -root η_r whose addition makes this a base of $\mathfrak{g}_{\text{fin}}$ with γ its corresponding fundamental coweight. Explicitly, η_r is the unique root with $\langle \gamma, \eta_r \rangle = 1$ such that $\eta_r - \eta_i$ is never a root.

Now fix such a choice $\eta_1, \dots, \eta_{r-1}$. Let g be a charge such that the roots sent to $\pi/2$ are exactly the linear combinations of η_r and δ , and for all $1 \leq i \leq r-1$, the positive roots in $p^{-1}(\eta_i)$ are $>_g \delta$. Then \mathfrak{g}_g is rank 2 affine, and thus is of type $A_1^{(1)}$ or $A_2^{(2)}$. Denote the simple roots of \mathfrak{g}_g (for the positive cone induced by that of g) by $\beta_{\underline{1}}$ and $\beta_{\underline{0}}$, where we take $\beta_{\underline{1}}$ to be the unique simple root whose image under π lies in the positive root cone of $\mathfrak{g}_{\text{fin}}$. For $i = 0, 1$, define $\ell_i = \frac{|\beta_i|}{\sqrt{2}}$ (which is always 1 or 2).

Definition 3.23 Let $\mathcal{L}_{\lambda, \gamma}$ be the element of the lowest weight crystal for \mathfrak{g}_g generated by the trivial module with Lusztig datum λ for the ordering $\beta_{\underline{1}} > \beta_{\underline{0}}$, as defined in [BDKT]. Explicitly, one easily show using the combinatorics in [BDKT] that

$$\mathcal{L}_{\lambda, \gamma} = \tilde{e}_{\underline{1}}^{\ell_1 \lambda_1} (\tilde{e}_{\underline{0}}^*)^{\ell_0 \lambda_1} (\tilde{e}_{\underline{1}}^*)^{\ell_1 \lambda_2} \tilde{e}_{\underline{0}}^{\ell_0 \lambda_2} \tilde{e}_{\underline{1}}^{\ell_1 \lambda_3} (\tilde{e}_{\underline{0}}^*)^{\ell_0 \lambda_3} \dots \mathcal{L}_{\emptyset}.$$

Lemma 3.24 If M is a simple module of weight $n\delta$ which is semi-cuspidal for g and in the crystal component of \mathcal{L}_{\emptyset} , and also semi-cuspidal for γ , then $M \cong \mathcal{L}_{\lambda, \gamma}$ for some λ .

Proof. As in the proof of Proposition 3.14, we can use Saito reflections to reduce to the case where $\beta_{\underline{0}}$ is a simple root.

Consider a representation M which is g -cuspidal of weight $n\delta$. Then M is of the form $\mathcal{L}_{\lambda, \gamma}$ if and only if its real Lusztig data $\mathfrak{a}_{(m+1)\beta_{\overline{0}} + m\beta_{\overline{1}}}(M)$ (as defined in Definition 2.20) with respect to the order $\beta_{\overline{1}} > \beta_{\overline{0}}$ is always trivial. Thus it suffices to prove that if M is semi-cuspidal for g and in the component of \mathcal{L}_{\emptyset} , and M has non-trivial Lusztig data of the form $\mathfrak{a}_{(m+1)\beta_{\overline{0}} + m\beta_{\overline{1}}}(M)$ for some $m \geq 0$, then M is not semi-cuspidal for c_{γ} .

We proceed by induction on the smallest integer m such that $\mathfrak{a}_{(m+1)\beta_{\underline{0}} + m\beta_{\underline{1}}}(M) \neq 0$, proving the statement for all γ simultaneously. If $m = 0$ the statement is clear, giving the base case of the induction.

So assume $m > 0$. Consider $\sigma_{\underline{0}}^* M$. By Corollary 2.18, this must be semi-cuspidal for the charge $g^{s_{\underline{0}}}$. The face-crystal for $g^{s_{\underline{0}}}$ is still rank-2 affine, with simple roots $\beta_{\underline{0}}$ and $\beta_{\underline{1}}$, and the Lusztig data of $\sigma_{\underline{0}}^* M$ for the order $\beta_{\underline{1}} < \beta_{\underline{0}}$ are given by $\bar{a}_{\alpha}(\sigma_{\underline{0}}^* M) = a_{s_{\underline{0}}\alpha}(M)$ for $\alpha \neq \beta_{\underline{0}}$. Note that

$$s_{\underline{0}}((m+1)\beta_{\underline{0}} + m\beta_{\underline{1}}) = (m-1)\beta_{\underline{0}} + m\beta_{\underline{1}}.$$

Since our inductive assumption covered all chamber weights, we are assuming that $\sigma_{\underline{0}}^* M$ is not semi-cuspidal for $c_{s_{\underline{0}}\gamma}$. But then applying Corollary 2.18 again it is clear that M is not semi-cuspidal for c_{γ} . This completes the proof. \square

Proposition 3.25 *The modules $\mathcal{L}_{\pi;\gamma}$ are a complete, irredundant list of lowest-weight semi-cuspidal modules of argument $\pi/2$ for c_γ , and this labeling is independent of the choice of base in \mathfrak{g}_{c_γ} .*

Proof. First, we assume that $\gamma = \omega_i^\vee$ is a fundamental coweight; in this case, we can take the η_i 's to be simple roots. Having made this assumption, the lowest-weight semi-cuspidal modules of argument $\pi/2$ for $c_{\omega_i^\vee}$ are precisely the semi-cuspidal modules of argument $\pi/2$ which are killed by \tilde{f}_j for $j \neq i, 0$.

Now, consider a representation L which is semi-cuspidal of argument $\pi/2$ and lowest-weight for $c_{\omega_i^\vee}$; we first wish to show that L is semi-cuspidal for g . If L is not semi-cuspidal for g , then there must be a g -cuspidal representation Q whose weight is a real root α such that L is a quotient of $Q \circ Q'$. Since $\alpha >_g \delta$, we must have that $p(\alpha)$ is a multiple of a positive root of $\mathfrak{g}_{\text{fin}}$, and $\alpha \geq_{c_{\omega_i^\vee}} \delta$. Since L is semi-cuspidal for $c_{\omega_i^\vee}$, we must have that $\alpha \leq_{c_{\omega_i^\vee}} \delta$, which implies that we must have $c_{\omega_i^\vee}(\alpha) \in i\mathbb{R}$, so Q is semi-cuspidal for $c_{\omega_i^\vee}$. It must be lowest weight for $\mathfrak{g}_{c_{\omega_i^\vee}}$; this is impossible if α is a real root by Corollary 3.18. Thus, we may conclude that L is semi-cuspidal for g .

As in Proposition 3.19, there exist canonical representations with M lowest-weight and N in the component of the identity for \mathfrak{g}_g such that $L = M \circ N = N \circ M$. Thus, we must have that M and N are both semi-cuspidal and lowest-weight for $\mathfrak{g}_{c_{\omega_i^\vee}}$. The representation M is killed by \tilde{f}_i since it is lowest-weight for \mathfrak{g}_g , by \tilde{f}_0 since it is semi-cuspidal for g and α_0 is the minimal root for this order, and by all other \tilde{f}_j 's since it is lowest-weight for $\mathfrak{g}_{c_{\omega_i^\vee}}$. Thus, $M = \mathcal{L}_0$, and $L = N$. Since L is semi-cuspidal for $c_{\omega_i^\vee}$, Lemma 3.24 implies that the Lusztig datum of L for the action of \mathfrak{g}_g ordering $\beta_1 > \beta_0$ must be purely imaginary. Since the number of lowest-weight cuspidals for $c_{\omega_i^\vee}$ of each weight is the same as the number of simples in the component of the trivial with imaginary Lusztig data, these sets of simples must coincide. This establishes the result for $\gamma = \omega_i^\vee$, with the obvious choice of base.

Now, we reduce all other cases to this one. If we still assume $\gamma = \omega_i^\vee$, but we had chosen a different base η'_i of $\mathfrak{g}_{\text{fin};\gamma}$, then we can find an element $w = s_{i_1} \cdots s_{i_k}$ of the Weyl group $W_{\text{fin};\gamma}$ such that $w\eta_i = \eta'_i$; applying the Saito reflections $S_{i_1} \cdots S_{i_k}$ to $\mathcal{L}_{\pi;\gamma}$ leaves this module unchanged (since $\mathcal{L}_{\pi;\gamma}$ is killed by \tilde{f}_{i_m} and $\tilde{f}_{i_m}^*$), and also sends this module as defined using $\{\eta_i\}$ to that defined using the $\{\eta'_i\}$. Thus $\mathcal{L}_{\pi;\gamma}$ is independent of this choice.

Now, consider a general chamber coweight γ ; if γ is not a fundamental coweight, then there must be a simple root α_i with $i > 0$ such that $\gamma(\alpha_i) < 0$. We must have $\alpha_i \neq \beta_1, \beta_0$ since $\gamma(\beta_1) = 0$ and $\gamma(\beta_0)$ can only be simple if it is equal to α_0 .

By Lemma 3.20, applying S_i to all cuspidal modules for c_γ defines an isomorphism of crystals to the same set-up for $s_i\gamma$, which is negative on one fewer positive root in the finite type system than γ ; furthermore, it sends $\mathcal{L}_{\pi;\gamma}$ to $\mathcal{L}_{\pi;s_i\gamma}$. By induction, we

may reduce to the case where γ is a fundamental coweight ω_j^\vee , which establishes the result. \square

Now fix a generic charge c such that $c(\delta) \in i\mathbb{R}_{>0}$. This defines a positive system in the finite type root system, where we say $\bar{\alpha}$ is positive if $p^{-1}(\alpha) >_c \delta$. Let $\bar{\chi}_1, \dots, \bar{\chi}_r$ be the corresponding set of simple roots and $\gamma_1, \dots, \gamma_r$ the dual set of coweights. For each r -tuples of partitions $\pi = (\pi^{\gamma_1}, \dots, \pi^{\gamma_r})$, define

$$(6) \quad \mathcal{L}(\pi) = \mathcal{L}_{\pi^{\gamma_1}; \gamma_1} \circ \mathcal{L}_{\pi^{\gamma_2}; \gamma_2} \circ \dots \circ \mathcal{L}_{\pi^{\gamma_r}; \gamma_r}.$$

Remark 3.26 These agree with Kleshchev's imaginary modules [Kle, §4.3]. Note that in contrast to Kleshchev, we have a *canonical* labeling of these by multipartitions.

Lemma 3.27 *The module $\mathcal{L}(\pi)$ is irreducible and independent of the ordering on simple roots. As π ranges over multipartitions with n boxes, these modules are all distinct and a complete list of c -semi-cuspidal representations of $R_{n\delta}$.*

Proof. We induct on the largest j such that $\pi^{\gamma_j} \neq \emptyset$. Note that for the face defined by the common maxima of γ_k for $1, \dots, j-1$, the module $\mathcal{L}_{\pi^{\gamma_1}; \gamma_1} \circ \mathcal{L}_{\pi^{\gamma_2}; \gamma_2} \circ \dots \circ \mathcal{L}_{\pi^{\gamma_{j-1}}; \gamma_{j-1}}$ is a lowest weight simple for the crystal of the face by the inductive hypothesis. Furthermore, by definition, $\mathcal{L}_{\pi^{\gamma_j}; \gamma_j}$ is in the component of the identity for the face crystal. Thus, $\mathcal{L}_{\pi^{\gamma_1}; \gamma_1} \circ \mathcal{L}_{\pi^{\gamma_2}; \gamma_2} \circ \dots \circ \mathcal{L}_{\pi^{\gamma_j}; \gamma_j}$ is irreducible by Proposition 3.19. The independence of ordering immediately follows from the irreducibility.

Clearly all the representations $\mathcal{L}(\pi)$ are semi-cuspidal. They are all distinct, since the partition π^{γ_i} is uniquely determined by the isomorphism type of $L(\pi)$. By Corollary 2.7, this is the right number of semi-cuspidal representations. \square

Thus, if we consider the KLR polytope \tilde{P}_L of L , the labeling of each real edge is determined by its length and direction; any imaginary edge parallel to δ could be labeled with any $\mathcal{L}(\pi)$ (where we fix a charge c is chosen so that the imaginary edge lies in its associated walk). However, we now show that this information can be encoded in a labeling of facets rather than edges.

Fix a simple L and a finite type chamber coweight γ . Consider the semi-cuspidal decomposition $(\dots, L_2, L_1, L_0, L^1, L^2, \dots)$ of L for c_γ where L_0 is the component with argument $\pi/2$.

Definition 3.28 *Let π^γ be the partition such that the representation L_0 lies in the crystal component of $\mathcal{L}_{\pi^\gamma; \gamma}$.*

Proposition 3.29 *The representation decorating any imaginary edge E is exactly $\mathcal{L}(\pi^{\gamma_1}, \dots, \pi^{\gamma_r})$, where the γ_i are the chamber coweights which achieve their minimal value on E , and π^γ is the partition associated to L and γ_i by Definition 3.28.*

Proof. Fix a representation L . Let c be a generic charge such that E is part of the path P_c^c . Let M be the representation in the semi-cuspidal decomposition of L for c whose weight is a multiple of δ . Then M is also semi-cuspidal for each c_γ (since the value of γ on the imaginary edge is a minimum on P_L). We need only show that if $M = \mathcal{L}(\xi^{\gamma_1}, \dots, \xi^{\gamma_r})$, then the partition π^{γ_i} attached to L by Definition 3.28 is ξ^{γ_i} . If c is a small deformation of c_γ , then is clear. Thus, we need only consider how π^γ changes as we deform c to c_γ linearly. While the semi-cuspidal decomposition changes, the imaginary edge that the associated walk goes along never changes. Thus, the representation in the semi-cuspidal decomposition of weight which is a multiple of δ never changes. Thus, that representation is always M . This establishes the result. \square

Thus, the KLR polytope in the sense of Definition 3.3 can be encoded as a decorated affine pseudo-Weyl polytope as defined in the introduction, where we decorate the facet where γ achieves its minimum with π^γ .

3.5. An example. If one were trying to naively generalize the finite type situation, it would be natural to hope that for a fixed generic charge, one could find a totally ordered set of cuspidal simples, with the number in each weight being the root multiplicity, such that, for $L_1 \leq \dots \leq L_k$ the module $A(L_1^{n_1}, \dots, L_k^{n_k})$ gives a complete list of the simples. In fact, even in affine type, this will fail, as we now illustrate. We should note that the same example is treated in [Kas, Example 3.3], but we wish to give a treatment emphasizing the features of interest to us.

The structure of this example depends on the polynomial $Q_{01}(u, v) = u^2 + quv + v^2$ for some $q \in \mathbb{k}$ (we note that this is not a completely general choice of Q , but any choice of Q gives an algebra isomorphic to this one after passing to a finite field extension) and thus provides an excellent example of sensitivity to this polynomial.

Now, consider the weight space for 2δ ; this is the easiest example of a root where there are *no* cuspidal representations. These representations can be described as $\mathcal{L}_{(2);\omega} = \tilde{e}_1^2 \tilde{e}_0^2 \mathcal{L}_0$ and $\mathcal{L}_{(1,1);\omega} = \tilde{e}_1 \tilde{e}_0 \tilde{e}_1 \tilde{e}_0 \mathcal{L}_0$. Now, consider the induction $\mathcal{L}_{(1);\omega} \circ \mathcal{L}_{(1);\omega}$. This space is 6-dimensional, spanned by the elements

$$\begin{array}{lll}
 v = \begin{array}{c} 0 \ 1 \ 0 \ 1 \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} & \psi_2 v = \begin{array}{c} 0 \ 0 \ 1 \ 1 \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} & \psi_3 \psi_2 v = \begin{array}{c} 0 \ 0 \ 1 \ 1 \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \\
 \psi_1 \psi_2 v = \begin{array}{c} 0 \ 0 \ 1 \ 1 \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} & \psi_3 \psi_1 \psi_2 v = \begin{array}{c} 0 \ 0 \ 1 \ 1 \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} & \psi_2 \psi_3 \psi_1 \psi_2 v = \begin{array}{c} 0 \ 1 \ 0 \ 1 \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array}
 \end{array}$$

where v is any non-zero element of $\mathcal{L}_{(1);\omega} \boxtimes \mathcal{L}_{(1);\omega}$, which is 1-dimensional.

The span H of the basis vectors other than v is a submodule (it is the kernel of a map to $\mathcal{L}_{(2);\omega}$). The image of the idempotent e_{0011} is irreducible over $R_{00} \otimes R_{11}$, and generates H . Thus, either

- H is irreducible or
- $\psi_2\psi_3\psi_1\psi_2v$ spans a submodule.

$$\psi_2^2\psi_3\psi_1\psi_2v = \begin{array}{c} 0 \quad 0 \quad 1 \quad 1 \\ \text{Diagram 1} \end{array} = q \begin{array}{c} 0 \quad 0 \quad 1 \quad 1 \\ \text{Diagram 2} \end{array} = -q \begin{array}{c} 0 \quad 0 \quad 1 \quad 1 \\ \text{Diagram 3} \end{array}$$

Thus, if $q \neq 0$, H is irreducible and thus $H \cong \mathcal{L}_{(2);\omega}$. Its inclusion is split, with complement spanned by $qv + \psi_2\psi_3\psi_1\psi_2v$. In particular, $\mathcal{L}_{(1);\omega} \circ \mathcal{L}_{(1);\omega}$ is semi-simple with both $\mathcal{L}_{(2);\omega}$ and $\mathcal{L}_{(1,1);\omega}$ occurring as summands. We see that neither of these modules can thus be cuspidal, since

$$\text{ch}(\mathcal{L}_{(2);\omega}) = (q^{-2} + 2 + q^2)w[0011] + w[0101].$$

If $q = 0$, then the behavior is quite different; in this case $\psi_2\psi_3\psi_1\psi_2v$ spans the socle of $\mathcal{L}_{(1);\omega} \circ \mathcal{L}_{(1);\omega}$, and H is its radical. In particular, $\mathcal{L}_{(1);\omega} \circ \mathcal{L}_{(1);\omega}$ is indecomposable, and a 3-step extension where a copy of $\mathcal{L}_{(2);\omega}$ is sandwiched between the socle and cosocle, both isomorphic to $\mathcal{L}_{(1,1);\omega}$. In particular, when $q = 0$, the representation $\mathcal{L}_{(2);\omega}$ is cuspidal, since

$$\text{ch}(\mathcal{L}_{(2);\omega}) = (q^{-2} + 2 + q^2)w[0011].$$

The KLR polytopes of these representations are independent of q and are given by



If one takes the choice of parameters as in [VV11] corresponding to an Ext-algebra of perverse sheaves on the moduli of representations of a Kronecker quiver (which is also that fixed by [BK09] in order to find a relationship to affine Hecke algebras either at $q = -1$ or in characteristic 2), then we take $q = -2$. Thus, if the field \mathbb{k} has characteristic $\neq 2$, we have $q \neq 0$ and $\dim \mathcal{L}_{(2);\omega} = 5$ whereas if \mathbb{k} does have characteristic 2, then $q = 0$ and $\dim \mathcal{L}_{(2);\omega} = 4$. Under Brundan and Kleshchev's isomorphism [BK09] between quotients of KLR algebras and cyclotomic Hecke algebras, this corresponds to the change in characters as we pass from the Hecke algebra at a root of unity to the symmetric group, or the difference between the canonical basis and 2-canonical basis. The same example is considered from other perspectives by Kashiwara in [Kas, Ex. 3.3] the second author in [Web, 5.6]

3.6. Proof of Theorem B and Theorem C. Fix any convex order $<$. For each affine Lusztig datum there can be at most one decorated polytope satisfying the conditions of Theorem B which has the specified Lusztig datum with respect to $<$. Furthermore, by Theorem 2.3 and Corollary 2.7 we can always find a simple L such that P_L has this Lusztig datum. Thus, to prove Theorem B it suffices to show that each P_L satisfies all the specified conditions on 2-faces.

Every 2-face is either real or parallel to δ . The real 2-faces satisfy the condition by Lemma 3.21. Thus it remains to check the specified conditions on 2-faces parallel to δ . Fix a charge c such that the roots sent to the imaginary line form a rank 2 sub-root system, and let \mathfrak{g}_c be the associated rank 2 affine algebra. This defines a (possibly degenerate) 2-face F_c of any P_L , and all 2 faces occur this way for a well-chosen c .

Let $\gamma_1, \dots, \gamma_{r-1}$ be the $r - 1$ finite type chamber weights which define faces of P_L containing F_c for all L , and γ_+, γ_- the two chamber weights that define faces that intersect F_c in vertical lines. If you deform c a small amount, then it gives a complete order on roots, and picks out one of the two vertical edges of F_c . We can choose deformations c_\pm such that the set of chamber weights associated with these charges are $\{\gamma_1, \dots, \gamma_{r-1}, \gamma_\pm\}$. Let $\beta_{\underline{0}}$ and $\beta_{\underline{1}}$ be the simple roots parallel to F_c with

$$\langle \gamma_+, \beta_{\underline{1}} \rangle > 0 > \langle \gamma_+, \beta_{\underline{0}} \rangle \quad \langle \gamma_-, \beta_{\underline{0}} \rangle > 0 > \langle \gamma_-, \beta_{\underline{1}} \rangle.$$

We use the notations $\mathcal{L}_\beta, \mathcal{L}_{\pi, \gamma_i}, \mathcal{L}(\pi)$ for the cuspidal representations for c_+ and $\bar{\mathcal{L}}_\beta, \bar{\mathcal{L}}_{\pi, \gamma_i}, \bar{\mathcal{L}}(\pi)$ for c_- . Similarly, we use a_β to denote the Lusztig data of a \mathfrak{g} -polytope with respect to c_+ or a \mathfrak{g}_c -polytope for the order where $\beta_{\underline{0}} > \beta_{\underline{1}}$ and \bar{a}_β for c_- or the order where $\beta_{\underline{1}} > \beta_{\underline{0}}$.

We define a map $L \mapsto P_L^F$ from the set of c -semi-cuspidal representations of weight parallel to F to decorated pseudo-Weyl polytopes for \mathfrak{g}_F by sending L to the polytope P_L^F with Lusztig data given by

$$a_\beta(P_L^F) = a_\beta(P_L) \quad \bar{a}_\beta(P_L^F) = \bar{a}_\beta(P_L)$$

for every real root β parallel to F and

$$(7) \quad a_\delta(P_L^F) = \pi^{\gamma^+} \quad \bar{a}_\delta(P_L^F) = \pi^{\gamma^-}.$$

Let L, M, N be as in the statement of Proposition 3.19 for the charge c .

Proposition 3.30 *The polytope P_L^F is the MV polytope (in the sense of [BDKT]) for the crystal element N .*

Proof. We must check the conditions of Theorem 1.22.

- (i) This is clear when L is a lowest weight object for the crystal (in which case the weight is 0 on both sides), and it is also clear that this property is preserved by the \mathfrak{g}_c crystal operators.

- (ii.1-4) Using Lemma 2.19, we can find Saito reflections in $B(-\infty)$ which reduce us to the case where $\beta_{\underline{0}}$ or $\beta_{\underline{1}}$ is simple for \mathfrak{g} . Hence this is a consequence of a consequence of Proposition 3.10 and the form of $*$ involution on $B(-\infty)$.
- (iii.1-4) Again, using Lemma 2.19, we can assume that $\beta_{\underline{0}}$ or $\beta_{\underline{1}}$ is simple in \mathfrak{g} . Assuming $\beta_{\underline{0}}$ is simple, it is clear that the Saito reflections in this root in $B^{\mathfrak{sc}}(-\infty)$ are the restrictions of the corresponding reflections in the full crystal $B(-\infty)$. Hence the statements for these two reflections are a consequence of Corollary 2.18. To get the statements for the reflections in $\beta_{\underline{1}}$ we instead use Saito reflections in $B(-\infty)$ to reduce this to a simple root.
- (iv) By definition, $\mathcal{L}_{\lambda;\gamma} = \tilde{\mathbf{e}}_1^{\ell_1\lambda_1}(\tilde{\mathbf{e}}_0^*)^{\ell_0\lambda_1}\tilde{\mathcal{L}}_{\lambda\setminus\lambda_1;\gamma}$. Since this crystal element has trivial real Lusztig data for $\beta_{\underline{1}} > \beta_{\underline{0}}$, we know that

$$\tilde{\mathbf{f}}_1^{\ell_1\lambda_1}\tilde{\mathcal{L}}_{\lambda\setminus\lambda_1;\gamma} = (\tilde{\mathbf{f}}_0^*)^{\ell_0\lambda_1}\tilde{\mathcal{L}}_{\lambda\setminus\lambda_1;\gamma} = 0.$$

Thus, we see that

$$\bar{a}_{\alpha_{\underline{1}}}(P_{\mathcal{L}_{\lambda;\gamma}}^F) = \ell_1\lambda_1 \quad \bar{a}_{\alpha_{\underline{0}}}(P_{\mathcal{L}_{\lambda;\gamma}}^F) = \ell_0\lambda_1.$$

Furthermore, the module $\tilde{\mathcal{L}}_{\lambda\setminus\lambda_1;\gamma}$ is semi-cuspidal of weight $(n - \lambda_1)\delta$ for the order $\beta_{\underline{1}} > \beta_{\underline{0}}$ and by the definition (7), this means that $\bar{a}_{\delta}(P_{\mathcal{L}_{\lambda;\gamma}}^F) = \lambda \setminus \lambda_1$. This establishes the final condition of Theorem 1.22. \square

This establishes Theorem B.

Proof of Theorem C. It follows from Lemmas 1.14 and 1.15 that the sets of decorated pseudo-Weyl polytopes P_L and HN_b are both uniquely determined by a single Lusztig datum and relations extending the tropical Plücker relations that determine the structure of 2-faces. Thus, the recursive nature of the relations for both polytopes means we only need to check that the KLR polytopes and HN polytopes coincide in the rank 2 case. For real 2-faces, this follows from the Theorem A and [BKT, §1.5] and for affine 2-faces, this follows from the match (with a transpose) of the $\widehat{\mathfrak{sl}}_2$ MV polytopes defined of [BKT] and [BDKT], which is shown in [MT]. Thus, every KLR polytope is an HN polytope with its imaginary Lusztig data transposed. This determines some bijection $B(-\infty) \rightarrow B(-\infty)$. This bijection preserves Lusztig data for every convex order, is thus a crystal isomorphism, and is therefore the identity. \square

3.7. Beyond affine type. In affine type, while we can have many different semi-cuspidal representations corresponding to an imaginary root, we still have considerable control over the structure of these representations, as the proceeding sections show. In particular, the additional structure in their polytopes can be captured in a straight-forward way by labeling facets with partitions.

In general, we expect that the structure of a 2-face should be controlled by the set of roots obtained by intersecting a 2-dimensional plane with Δ . If \mathfrak{g} is of finite type then this set is also a finite type root system and the 2-faces are finite type MV polytopes. In affine type, this intersection is a rank-2 affine root system, and 2-faces

are essentially rank 2 affine MV polytopes. But because of the multiplicities, the sum of these root spaces is actually not an affine Lie algebra—rather, it is an infinite-rank Borchers algebra whose Cartan matrix is obtained by adding infinitely many rows and columns of zeroes to the rank 2 affine matrix. The structure we have observed in the 2-faces, a crystal for the affine Lie algebra together with an infinite family of commuting operators seems, in fact, to be a manifestation of this larger algebra, as opposed to just the affine one.

Beyond affine type, when one intersects Δ with a 2-plane, the resulting set of real roots will generate a root system of rank at most 2. However, if there is to be a generalization of Theorem B, it is probably necessary to consider not just this rank 2 root system, but rather the entire sum of the root spaces; by [Bor95, Theorem 1], this will always be a Borchers algebra. The corresponding Cartan matrix may have many non-zero entries outside the Cartan matrix of the root system generated by the real roots. Nonetheless, one could hope to define MV polytopes for this algebra, and that the 2-faces could be matched to these. Unfortunately, even if this were possible, “reduction to rank two” would mean reduction to a Borchers algebra of possibly infinite rank, leaving it debatable whether this actually improves matters; it still may shed some rather interesting light on the structure of KLR algebras and their simple representations. Some cases, such as toroidal algebras (where the Cartan matrix remains positive semi-definite) may be more tractable.

To illustrate some of the difficulties, consider the Cartan matrix

$$(8) \quad \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

This is of hyperbolic type, and the imaginary root $\beta = \alpha_1 + \alpha_2 + \alpha_3$ has multiplicity 2. Fix a charge c with $c(\alpha_0) = 1 + i$, $c(\alpha_1) = -1 + i$, $c(\alpha_2) = i$. The only real root with $c(\alpha) \in i\mathbb{R}$ is α_2 itself. Thus the real roots only generate a copy of \mathfrak{sl}_2 , which is already a new phenomenon as in finite and affine type the real roots corresponding to a 2-face always generated a rank 2 root system.

Nonetheless, Proposition 3.14 shows that the semi-cuspidals of argument $\pi/2$ are a combinatorial crystal for \mathfrak{sl}_2 . If the analogue of Corollary 3.18 held, then we would have that \tilde{e}_2 and \tilde{e}_2^* were identical acting on every semi-cuspidal of argument $\pi/2$, since this is the case in $B_{\mathfrak{sl}_2}(-\infty)$. However, both $\tilde{e}_2\tilde{e}_1\tilde{e}_0\mathcal{L}_0$ and $\tilde{e}_2^*\tilde{e}_1\tilde{e}_0\mathcal{L}_0$ are 1-dimensional; the former has character $w[012]$ and the latter $w[201]$. Thus, they are necessarily distinct.

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